Construction of m-ovoids of $Q^+(7, q)$ with q odd

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joint work with Sam Adriaensen, Jan De Beule and Jonathan Mannaert

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A σ-sesquilinear form β is **non-degenerate** if it has the following property:

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For any k-dimensional subspace $S = \langle P_0, P_1, \ldots, P_k \rangle$ of $\mathrm{PG}(d, \mathbb{F})$, we define the subspace

$$S^{\perp} = P_0^{\perp} \cap P_1^{\perp} \cap \ldots \cap P_k^{\perp}$$

having dimension $d - k - 1$.

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Two maximum totally isotropic subspaces of $\mathrm{PG}(d, q)$ have the same dimension $r - 1$ and the integer r is called **rank of $\mathcal{P}$**.

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- **elliptic quadric** of $\mathrm{PG}(2r+1,q)$, $e=2$,

$$Q^-(2r+1,q) : X_0X_1 + \ldots + X_{2r-2}X_{2r-1} + f(X_{2r}, X_{2r+1}) = 0$$

  where f is a homogeneous irreducible polynomial of degree 2 over $\mathbb{F}_q$.

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- **parabolic quadric** of $\mathrm{PG}(2r, q)$, $e = 1$,

$$Q(2r, q) : X_0X_1 + \ldots + X_{2r-2}X_{2r-1} + X_{2r}^2 = 0.$$

- **hyperbolic quadric** of $\mathrm{PG}(2r-1,q)$, $e=0$,

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- **symplectic polar space** $W(2r-1,q)$ of $\mathrm{PG}(2r-1,q)$, $e=1$, defined by the following canonical reflexive, non-degenerate bilinear form

$$\beta(u,v) = u_0v_1 - u_1v_0 + \ldots + u_{2r-2}v_{2r-1} - u_{2r-1}v_{2r-2}.$$

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- **Hermitian variety** of $\mathrm{PG}(d,q^2)$

$$H(d,q^2): X_0^{q+1} + \ldots + X_d^{q+1} = 0$$

with $(d,e) \in \{(2r-1,1/2),(2r,3/2)\}$.

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The complement of an m-ovoid is a $\left(\frac{q^r-1}{q-1} - m\right)$-ovoid.

Intersection patterns for m-ovoids of $Q^-(2n+1, q)$

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Proposition (Adriaensen, De Beule, G., Mannaert - Preprint)

Suppose that $\mathcal{O}$ is an m-ovoid of $Q^-(2n+1,q)$, with $m \geq 1$. Then any elliptic quadric $Q^-(2n-1,q) \subset Q^-(2n+1,q)$ meets $\mathcal{O}$ in either

$$(m-2)q^{n-1}+m \quad \text{or} \quad (m-1)q^{n-1}+m \quad \text{or} \quad mq^{n-1}+m$$

points, and all of these cases occur.

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An m-ovoid $\mathcal{O}$ of $Q^-(2n+1,q)$, $m \geq 1$, has non-empty intersection with any $Q^-(2n-1,q) \subset Q^-(2n+1,q)$.

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Proposition (Adriaensen, De Beule, G., Mannaert - Preprint)

Assume that q is odd. Consider $Q^+(7,q)$ and suppose that π_1 and π_2 are 5-dimensional subspaces in $\mathrm{PG}(7,q)$ such that $\dim(\pi_1 \cap \pi_2) = 3$. Suppose both π_1 and π_2 intersect $Q^+(7,q)$ in an elliptic quadric $Q^-(5,q)$, say Q_1 and Q_2 respectively. Then there exists a collineation Φ of $\mathrm{PG}(7,q)$ such that Q_1 is mapped into Q_2 and the set $Q_1 \cap Q_2$ is pointwised fixed.

Theorem (Adriaensen, De Beule, G., Mannaert - Preprint)

There exist $(q+1)$-ovoids in $Q^+(7,q)$, q odd, which are the union of two $(q+1)/2$-ovoids contained in distinct elliptic quadrics $Q^-(5,q)$.

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Now let Φ be a collineation of $\mathrm{PG}(7,q)$ that maps Q_1 into Q_2 and that fixes $Q_1 \cap Q_2$ pointwise.

Theorem (Adriaensen, De Beule, G., Mannaert - Preprint)

There exist $(q+1)$-ovoids in $Q^+(7,q)$, q odd, which are the union of two $(q+1)/2$-ovoids contained in distinct elliptic quadrics $Q^-(5,q)$.

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Consider $Q^-(3,q)$ in the ambient space $\mathrm{PG}(3,q)$. If $q=3$, $q=27$ or $q \equiv 1 \pmod 4$, then there exists a line spread of $\mathrm{PG}(3,q)$ containing $(q^2+1)/2$ 2-secant lines to $Q^-(3,q)$.

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Lemma (Adriaensen, De Beule, G., Mannaert - Preprint)

Consider the elliptic quadric $Q^-(5,3)$ and let S be a 3-dimensional subspace meeting $Q^-(5,3)$ in an elliptic quadric $Q^-(3,3)$. Then for any two distinct points $P, R \in S$, there exists a 2-ovoid $\mathcal{O}$ of $Q^-(5,3)$ with $\mathcal{O} \cap S = \{P, R\}$.

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Remark

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- *If* $q > 5$, *the same technique does not work, since any* $(q+1)/2$*-ovoid of* $Q^-(5,q)$ *meets any* $Q^-(3,q) \subset Q^-(5,q)$ *in more then* $|Q^-(3,q)|/3 = (q^2+1)/3$ *points.*

Thanks for your attention!