On the equivalence issue of a class of 2-dimensional linear maximum rank-metric codes

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Finite Geometry and Friends

Joint work with G. Longobardi, R. Trombetti

OUTLINE

Maximum Rank Metric Codes

Rank Metric Codes

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For our purpose, we will be discussing the linearized polynomial setting for RD-codes.
Just to fix the notation

- $q = p^r$ and $r, n \in \mathbb{Z}^+$;
- $\mathbb{F}_{q^n}$ Galois field with q^n elements.

RD-codes in linearized polynomials setting

$$\mathcal{L}_{n,q}[x] = \left\{ \sum_{i=0}^{k} c_i x^{q^i} : c_i \in \mathbb{F}_{q^n},\ k \in \mathbb{N} \right\}$$

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$\mathcal{C}$ **linear** RD code $\leftrightarrow$ $\mathcal{C}$ $\mathbb{F}_q$-**subspace** of $\tilde{\mathcal{L}}_{n,q}[x]$

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Linear sets on the projective line
Introduction

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$$|L| = \frac{q^k - 1}{q - 1}$$ L *is* **scattered**

$k = n$ L *is* **maximum scattered**

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L_U *is* **PΓL-equivalent** to L_W $\Leftrightarrow \exists\, \varphi \in \mathrm{P\Gamma L}(2, q^n)$ s.t. $L_U^\varphi = L_W$

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$\mathcal{C}_f$ is a linear MRD-code with minimum distance $d = n - 1$ if and only if L_f is a maximum scattered linear set of $\mathrm{PG}(1, q^n)$.

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Example. [B. Csajbók, Zanella, 2018]

Let $f(x) = x^q$ and $g(x) = x^{q^s}$ with $(s, n) = 1$ and $s \not\equiv \pm 1 \pmod{n}$.

Linear sets and codes

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$f(x)$ and $g(x)$ are not ΓL-equivalent, i.e., C_f is not equivalent to C_g.
But we can easily see $L_f = \{\langle (1, x^{q-1}) \rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n} \setminus \{0\}\} = L_g$.

Known families of MRD codes

KNOWN FAMILIES

$\mathcal{G}_{2,s} = \langle x, x^{q^s} \rangle_{\mathbb{F}_{q^n}}$, $1 \leq s \leq n-1$, $\gcd(s,n)=1$	[BL, 2000]
$\mathcal{H}_{2,s,\eta} = \langle x, \eta x^{q^s} + x^{q^{(n-1)s}} \rangle_{\mathbb{F}_{q^n}}$, $n \geq 4$, $\mathrm{N}_{q^n/q}(\eta) \notin \{0,1\}$, $\gcd(s,n)=1$	[Sheekey, 2016] for $s=1$[LP, 2001]
$\mathcal{K}_{n,s,\delta} = \langle x, \delta x^{q^s} + x^{q^{s+n/2}} \rangle_{\mathbb{F}_{q^n}}$, $n \in \{6,8\}$, $\gcd(s,n/2)=1$ $\mathrm{N}_{q^n/q^{n/2}}(\delta) \notin \{0,1\}$, for some conditions on δ and q	[CsMPZa, 2018]
$\mathcal{Z}_{6,\eta} = \langle x, x^q + x^{q^3} + \eta x^{q^5} \rangle_{\mathbb{F}_{q^6}}$, where $\eta \in \mathbb{F}_{q^6}^*$ such that $\eta^2 + \eta = 1$;	[CsMZ, 2018] (q odd, for $q \equiv 0, \pm 1 \pmod 5$) [MMZ, 2020] (for remaining congruences of q)
$\mathcal{Z}_{6,\zeta} = \langle x, x^q + x^{q^3} + \eta x^{q^5} \rangle_{\mathbb{F}_{q^6}}$, with q even and some conditions over $\eta \in \mathbb{F}_{q^6}^*$	[BLMT. 202x]
$\mathcal{C}_t = \langle x, x^q + x^{q^{t-1}} - x^{q^{t+1}} + x^{q^{2t-1}} \rangle_{\mathbb{F}_{q^n}}$, q odd, $n=2t$ with either $t \geq 3$ odd and $q \equiv 1 \pmod 4$, or t even	[LZa, 2021]
$\mathcal{C}_{h,t} = \langle x, x^q + x^{q^{t-1}} - h^{1-q^{t+1}} x^{q^{t+1}} + h^{1-q^{2t-1}} x^{q^{2t-1}} \rangle_{\mathbb{F}_{q^n}}$, where q odd, $n=2t$, $h \in \mathbb{F}_{q^{2t}} \setminus \mathbb{F}_{q^t}$ such that $\mathrm{N}_{q^{2t}/q^t}(h) = -1$	[LMTZ, 2022]
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Some tools for our work

Definition. [Longobardi and Zanella, 2023]

A q-polynomial $F(x) = \sum_{i=0}^{n-1} c_i x^{q^i}$ is in *standard form*, if the greatest common divisor m_F of the set of integers

$$\{(i-j) \pmod{n} : c_i c_j \neq 0 \text{ with } i \neq j\} \cup \{n\},$$

is strictly larger than 1. If this is the case, then $F(x)$ has the following fashion:

$$F(x) = \sum_{j=0}^{n/m-1} c_j x^{q^{s+jm}},$$

where $m = m_F$ and $0 \leq s < m_F$.

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Theorem. [G., Longobardi and Trombetti, 202x]

Let $\mathcal{C}_{F_i}$, $i = 1, 2$ be two 2-dimensional MRD codes where F_i, $i = 1, 2$, are scattered polynomials having the standard form. Then, they are equivalent if and only if there exists $a, b, c, d \in \mathbb{F}_{q^n}^*$ such that

$$dF_2(x) = F_1^{\rho}(ax) \quad \text{or} \quad F_1^{\rho}(bF_2(x)) = cx$$

for some $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$. In particular,

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Equivalence issue

THE CODES WE STUDY

Recently, a Class of 2 dimensional MRD codes over $\mathbb{F}_{q^n}^2$, $n = 2t$, with minimum distance $2t - 1$ MRD codes was extended by Neri, Santonastaso, and Zullo from a class of codes given by Longobardi, Marino, Trombetti, and Zhou.

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The codes are given by

$$\mathcal{C}_{h,t,s} = \langle x, \psi_{h,t,s}(x) \rangle_{\mathbb{F}_{q^{2t}}},$$

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- any code $\mathcal{C}_{h,t,s}$ with $t \geq 4$ and even, is already expressed by means of a polynomial in standard form.
- for $t = 3$ code $\mathcal{C}_H$ is an MRD code equivalent to $\mathcal{C}_{h,3,s}$, such that $H(x)$ is in standard form and

$$H(x) := H_{h,s}(x) = (1 - h^{1+q^{2s}})x^{q^s} + (h + h^2)x^{q^{3s}} + h^{1+q^{2s}}(h + h^{q^s})x^{q^{5s}} \in \tilde{\mathcal{L}}_{6,q}[x].$$

[Longobardi and Zanella, 2023]

THEOREM. [Neri, Santonastaso, Zullo, 2022]

Let $t \geq 5$ and consider $\mathcal{C}_{h,t,s}$ and $\mathcal{C}_{k,t,\ell}$ such that $(s, n) = 1 = (\ell, n)$. Then the codes $\mathcal{C}_{h,t,s}$ and $\mathcal{C}_{k,t,\ell}$ are equivalent if and only if one of the following collection of conditions are satisfied:

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$$\rho(h) = \begin{cases} \pm k, & \text{if } t \not\equiv 2 \pmod{4} \\ \lambda k, \text{ where } \lambda^{q^2+1} = 1 & \text{if } t \equiv 2 \pmod{4} \end{cases}$$

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EQUIVALENCE WITHIN THE CLASS $\mathcal{C}_{h,t,s}$

Theorem. [G., Longobardi, Trombetti, 2023]

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We have for $t = 3$, $r \notin \{2, 3, 4\}$ and for $t = 4$, $r \notin \{2, 4, 6\}$.

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We have for $t = 3$, $r \notin \{2, 3, 4\}$ and for $t = 4$, $r \notin \{2, 4, 6\}$.
Therefore we see for $t \in \{3, 4\}$ and $r \neq \pm 1$ or $r \neq t \pm 1$, then $\mathcal{C}_{h,t,s}$ and $\mathcal{C}_{k,t,\ell}$ can not be equivalent. The result follows from the following two lemmas.

On the equivalence issue in $t \in \{3, 4\}$

Lemma. [G., Longobardi, Trombetti, 202x]

Assume $n = 6$ and that $\mathcal{C}_{h,3,s}$ and $\mathcal{C}_{k,3,\ell}$, are equivalent. One gets the following:

1. if $\ell \equiv s \pmod 6$, then $h^\rho = \pm k$,
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$\Rightarrow$ It is sufficient to study these two cases, i.e.,

$$dH_{k,\ell}(x) = H^{\rho}_{h,s}(ax), \quad \text{and} \quad H^{\rho}_{h,s}(bH_{k,\ell}(x)) = cx$$

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MAIN RESULTS

Lemma. [G., Longobardi, Trombetti, 2023]

Assume $n = 6$ and that $\mathcal{C}_{h,3,s}$ and $\mathcal{C}_{k,3,\ell}$, are equivalent. One gets the following:

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Assume $n = 8$ and that $\mathcal{C}_{h,4,s}$ and $\mathcal{C}_{k,4,\ell}$, are equivalent. One gets the following:

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3. if $\ell \equiv 3s \pmod 8$, then $h^\rho = \pm k$,
4. if $\ell \equiv 5s \pmod 8$, then $h^\rho = \pm k^{-1}$,

where $\rho \in \mathrm{Aut}(\mathbb{F}_{q^8})$.

EQUIVALENCE ISSUE OF LINEAR SETS

Theorem. [Bartoli, Zanella, Zullo, 2020]

If $h \in \mathbb{F}_{q^2}$, the linear set $L_{h,3,s} \subset \mathrm{PG}(1, q^6)$ is PΓL-equivalent to some

$$L_\zeta = \{\langle(x, x^q + x^{q^3} + \zeta x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}^*_{q^6}\}$$

where $\xi \in \mathbb{F}_{q^6}$ such that $\xi^2 + \xi = 1$ if and only if $h \in \mathbb{F}_q$ and q is a power of 5. If $h \notin \mathbb{F}_{q^2}$, then $L_{h,3,s}$ is not PΓL-equivalent to $L_{2,s}$, $L_{2,s,\eta}$ and $L_{6,s,\delta}$ in $\mathrm{PG}(1, q^6)$.

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For $n = 8$, to study the equivalence issue of our codes we need to see its geometrical counterpart, i.e., maximum linear sets.

Equivalence issue of linear sets

Definition. [Zanella and Zullo, 2020]

Let Γ be a subspace of $\mathrm{PG}(n-1, q^n)$ of dimension $k \geq 0$ such that $\Gamma \cap \Sigma = \emptyset$ and $\dim(\Gamma \cap \Gamma^\sigma) \geq k-2$. Let r be the minimum positive integer such that

$$\dim(\Gamma \cap \Gamma^{\hat\sigma} \cap \cdots \cap \Gamma^{\hat\sigma^\gamma}) > k - 2\gamma.$$

The integer γ is called the *intersection number of* Γ *w.r.t* $\hat\sigma$ *and is denoted by* $\mathrm{intn}_\sigma(\Gamma)$.

We observe that for if $n = 8$, if L be a maximum scattered linear set of Λ in $\mathrm{PG}(7, q^8)$ and $\mathrm{intn}_\sigma(\Gamma)$ does not belong to $\{1, 2\}$, then L is neither equivalent to $L_{2,s}$ nor to $L_{2,s,\eta}$. (Similar statement appears for $n = 6$ in [Bartoli, Zanella, Zullo, 2020])

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Let $h^{1+q^{4s}} = -1$ and $(s, 4) = 1$. The linear set $L_{h,4,s}$ can be seen as the projection of the canonical subgeometry Σ from the vertex

$$\Gamma_s : \begin{cases} x_0 = 0 \\ x_s + x_{3s} - h^{1-q^{5s}} x_{5s} + h^{1-q^{7s}} x_{7s} = 0 \end{cases}$$

onto the line(axis) $\mathrm{PG}(1, q^8) \subset \mathrm{PG}(7, q^8)$ with equations $x_{2s} = x_{3s} = \cdots = x_{7s} = 0$, where indices in system above are taken modulo 8.

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where $u \in \{s, 3s, 5s, 7s\} \pmod 8$.

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where $u \in \{s, 3s, 5s, 7s\} \pmod 8$. Then $\dim(\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u}) = 3$ and $\dim(\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u} \cap \Gamma_s^{\hat{\sigma}^{2u}}) = 1$ for any $u \in \{s, 3s, 5s, 7s\} \pmod 8$.

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INEQUIVALENCE WITH OTHER CODES

Theorem. [G., Longobardi, Trombetti, 2023]

Let $n = 2t$, $t \in \{3, 4\}$, $h, k, \eta \in \mathbb{F}_{q^n}$ satisfying $\mathrm{N}_{q^n/q^t}(h) = \mathrm{N}_{q^n/q^t}(k) = -1$ and $\mathrm{N}_{q^n/q}(\eta) \neq 1$. Let $s \in \mathbb{N}$ such that $(n, s) = 1$.

(a) $\mathcal{H}_{2,s}(\eta)$ and $\mathcal{C}_{h,t,s}$ are not equivalent.

(b) Assume $\delta \in \mathbb{F}_{q^{2t}}$ such that $\mathrm{N}_{q^n/q^{n/2}}(\delta) \notin \{0, 1\}$, and the other conditions on δ and q as expressed in [Csajbok, Marino, Polverino, Zanella, 2018], hold true. Then, the codes $\mathcal{K}_{2t,s,\delta}$ and $\mathcal{C}_{h,t,s}$ are not equivalent.

(c) Assume $\zeta \in \mathbb{F}_{q^6}$ such that $\zeta^2 + \zeta = 1$. Then, the codes $\mathcal{Z}_{6,\zeta}$ and $\mathcal{C}_{h,3,s}$ are not equivalent except $h \in \mathbb{F}_q$ and q a power of 5.

That's all for today!