Finite Geometry & Friends

A Brussels' Summer School on Finite Geometry

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Decoding Interleaved Linearized Reed–Solomon Codes

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joint work with Hannes Bartz[1] and Sven Puchinger[3]

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DLR

Outline

1. Motivation from Code-Based Cryptography
2. Interleaving in the Sum-Rank Metric
3. Interleaved Linearized Reed–Solomon Codes
4. Implications for Code-Based Cryptography

Code-Based Cryptography

- quantum computers can solve today's cryptographic primitives efficiently

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  - three remaining **code-based** KEM candidates

- code-based cryptography suffers from large key sizes
  $\Longrightarrow$ many approaches to mitigate this issue are studied,
  e.g. alternative metrics or codes with high error-correction capability

McEliece Cryptosystem [McEliece, 1978]

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public key

- generator matrix $\boldsymbol{G}_{pub} \in \mathbb{F}_{q^m}^{k \times n}$ of $\mathcal{C}$ that does not reveal an efficient decoder
- error weight t

private key

- efficient decoder for $\mathcal{C}$ with decoding radius at least t

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Alice | Bob

$$\boldsymbol{x} \in \mathbb{F}_{q^m}^k \xrightarrow{\text{encrypt}} \boldsymbol{x}\boldsymbol{G}_{pub} + \boldsymbol{e} \xrightarrow{\text{transmit}} \boldsymbol{x}\boldsymbol{G}_{pub} + \boldsymbol{e} \xrightarrow{\text{decrypt}} \boldsymbol{x} \in \mathbb{F}_{q^m}^k$$

random $\boldsymbol{e} \in \mathbb{F}_{q^m}^n$
of weight t

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Some Weights

Consider $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}_{q^m}^n$ and define

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  where $\mathrm{rk}_q(\boldsymbol{x})$ is the maximum number of $\mathbb{F}_q$-linearly independent entries of $\boldsymbol{x}$,
- and its **sum-rank weight** (with respect to a fixed length partition)

$$\mathrm{wt}_{\Sigma R}(\boldsymbol{x}) := \sum_{i=1}^{\ell} \mathrm{rk}_q\left(\boldsymbol{x}^{(i)}\right) = \mathrm{rk}_q\left(\boldsymbol{x}^{(1)}\right) + \mathrm{rk}_q\left(\boldsymbol{x}^{(2)}\right) + \cdots + \mathrm{rk}_q\left(\boldsymbol{x}^{(\ell)}\right).$$

Some Metrics

The **metrics** for $\star \in \{H, rk, \Sigma R\}$ are $d_\star(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{wt}_\star(\boldsymbol{x} - \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}_{q^m}^n$.

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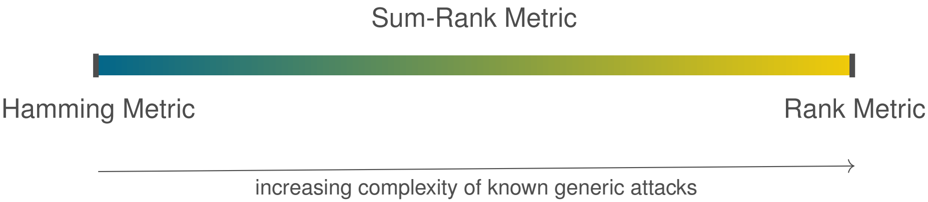

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Sum-Rank Metric

Hamming Metric

Rank Metric

increasing complexity of known generic attacks

structural attacks
broke many proposals

Interleaved Codes

For a linear code $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$ and an interleaving order $s \in \mathbb{N}^*$, define

- the **vertically interleaved code**

$$\mathrm{VInt}(\mathcal{C}, s) := \left\{ \boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 \\ \vdots \\ \boldsymbol{c}_\ell \end{pmatrix} : \boldsymbol{c}_j \in \mathcal{C} \text{ for all } j = 1, \dots, s \right\} \subseteq \mathbb{F}_{q^m}^{s \times n},$$

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- and the **horizontally interleaved code**

$$\mathrm{HInt}(\mathcal{C}, s) := \left\{ \boldsymbol{c} = (\boldsymbol{c}_1 \mid \cdots \mid \boldsymbol{c}_\ell) : \boldsymbol{c}_j \in \mathcal{C} \text{ for all } j = 1, \dots, s \right\} \subseteq \mathbb{F}_{q^m}^{sn}.$$

Sum-Rank Weight of Vertically Interleaved Vectors

Choose an interleaving order $s \in \mathbb{N}^*$.
For $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s \in \mathbb{F}_{q^m}^n$, consider the matrix $\boldsymbol{X} \in \mathbb{F}_{q^m}^{s \times n}$ with

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_s \end{pmatrix} = \begin{pmatrix} \boxed{\boldsymbol{x}_1^{(1)} \ \boldsymbol{x}_1^{(2)} \ \cdots \ \boldsymbol{x}_1^{(\ell)}} \\ \boxed{\boldsymbol{x}_2^{(1)} \ \boldsymbol{x}_2^{(2)} \ \cdots \ \boldsymbol{x}_2^{(\ell)}} \\ \vdots \\ \boxed{\boldsymbol{x}_s^{(1)} \ \boldsymbol{x}_s^{(2)} \ \cdots \ \boldsymbol{x}_s^{(\ell)}} \end{pmatrix}.$$

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The sum-rank weight of $\boldsymbol{X}$ is

$$\mathrm{wt}_{\Sigma R}(\boldsymbol{X}) = \mathrm{rk}_q \begin{pmatrix} \boldsymbol{x}_1^{(1)} \\ \boldsymbol{x}_2^{(1)} \\ \vdots \\ \boldsymbol{x}_s^{(1)} \end{pmatrix} + \mathrm{rk}_q \begin{pmatrix} \boldsymbol{x}_1^{(2)} \\ \boldsymbol{x}_2^{(2)} \\ \vdots \\ \boldsymbol{x}_s^{(2)} \end{pmatrix} + \dots + \mathrm{rk}_q \begin{pmatrix} \boldsymbol{x}_1^{(\ell)} \\ \boldsymbol{x}_2^{(\ell)} \\ \vdots \\ \boldsymbol{x}_s^{(\ell)} \end{pmatrix}.$$

Sum-Rank Weight of Horizontally Interleaved Vectors

Choose an interleaving order $s \in \mathbb{N}^*$.
For $\boldsymbol{x}_1, \dots, \boldsymbol{x}_s \in \mathbb{F}_{q^m}^n$, consider the vector $\boldsymbol{x} \in \mathbb{F}_{q^m}^{sn}$ with

$$\begin{aligned}\boldsymbol{x} &= (\boldsymbol{x}_1 \mid \boldsymbol{x}_2 \mid \cdots \mid \boldsymbol{x}_s) \\ &= \left(\boxed{\boldsymbol{x}_1^{(1)} \; \boldsymbol{x}_1^{(2)} \cdots \boldsymbol{x}_1^{(\ell)}} \; \boxed{\boldsymbol{x}_2^{(1)} \; \boldsymbol{x}_2^{(2)} \cdots \boldsymbol{x}_2^{(\ell)}} \cdots \boxed{\boldsymbol{x}_s^{(1)} \; \boldsymbol{x}_s^{(2)} \cdots \boldsymbol{x}_s^{(\ell)}} \right).\end{aligned}$$

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Linearized Reed–Solomon Codes

Choose

- **code locators** $\beta = (\beta^{(1)} \mid \cdots \mid \beta^{(\ell)}) \in \mathbb{F}_{q^m}^n$ whose blocks $\beta^{(i)}$ contain $\mathbb{F}_q$-linearly independent elements
- and **evaluation parameters** $\xi = (\xi_1, \dots, \xi_\ell) \in \mathbb{F}_{q^m}^\ell$ belonging to pairwise distinct nontrivial θ-conjugacy classes of $\mathbb{F}_{q^m}$ (i.e., $\forall i_1 \neq i_2 \ \nexists c \in \mathbb{F}_{q^m}^* : \theta(c)\xi_{i_1} c^{-1} = \xi_{i_2}$).

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We define the **linearized Reed–Solomon (LRS) code** with these parameters as

$$\mathrm{LRS}[\beta, \xi; n, k] = \left\{ f(\beta)_\xi = (f(\beta^{(1)})_{\xi_1} \mid \cdots \mid f(\beta^{(\ell)})_{\xi_\ell}) : f \in \mathbb{F}_{q^m}[x; \theta]_{<k} \right\} \subseteq \mathbb{F}_{q^m}^n.$$

Interleaved Linearized Reed–Solomon Codes

For an LRS code $\mathcal{C} := \mathrm{LRS}[\boldsymbol{\beta}, \boldsymbol{\xi}; \boldsymbol{n}, k]$ and $s \in \mathbb{N}^*$, we consider

- the **vertically interleaved LRS (VILRS) code** $\mathsf{VInt}(\mathcal{C}, s)$

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We consider probabilistic-unique decoding up to an error weight

$$t \leq \frac{s}{s+1}(n-k).$$

$$\boxed{\begin{array}{c}\text{codeword}\\ \boldsymbol{c}\end{array}} \xrightarrow[\text{of weight } t]{\text{error } \boldsymbol{e}} \boxed{\begin{array}{c}\text{received word}\\ \boldsymbol{y} = \boldsymbol{c} + \boldsymbol{e}\end{array}}$$

Decoders for Interleaved LRS Codes

Decoders for VILRS Codes

- Loidreau–Overbeck-like
  [Bartz and Puchinger, 2023]
- interpolation-based
  [Bartz and Puchinger, 2023]
- syndrome-based (error-erasure)
  [Hörmann et al., 2023]

Decoders for HILRS Codes

- syndrome-based (error-erasure)
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- Gao-like
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- Loidreau–Overbeck-like
  $\mathcal{O}(sn^{\omega}) \subseteq \mathcal{O}(sn^{2.373})$
- interpolation-based
  $\tilde{\mathcal{O}}(s^{\omega}\mathcal{M}(n)) \subseteq \tilde{\mathcal{O}}(s^{2.373}n^{1.635})$
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$\tilde{\mathcal{O}}$ neglects logarithmic factors, $\omega < 2.373$ is the matrix-multiplication coefficient, and $\mathcal{M}(n) \subseteq \mathcal{O}(n^{1.635})$ denotes the cost of multiplication of two degree-n skew polynomials.

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  - subquadratic decoding complexity
- VILRS codes can only provide security for low interleaving order $s < t$ [Jerkovits et al., 2023]
- (more) investigation of potential attacks needed

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