The Twisted $D_{5,5}(q)$ Graph

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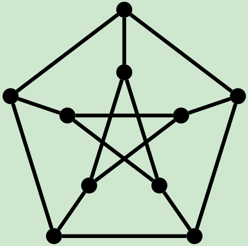
20 September 2023

Strongly Regular Graphs (SRGs)

A k-regular graph of order v is called **strongly regular** with parameters (v, k, λ, μ) (where $0 < k < v - 1$) if

1. any two adjacent vertices have precisely λ **common neighbors**,
2. any two nonadjacent vertices have precisely μ **common neighbors**.

Example (The Petersen Graph $(10, 3, 0, 1)$)



Grassmann Graphs

$V(n,q)$ = vector space of dimension n over field with q elements.

The Grassmann Graph $J_q(5,2)$.

This generalizes to $J_q(n,k)$.

Vertices: 2-spaces of $V(5,q)$.

Adjacency: meet in an 1-space.

$J_q(n,2)$ is an SRG.

In general, $J_q(n,k)$ is a distance-regular graph.

Twisted Grassmann Graphs I

$V(n,q)$ = vector space of dimension n over field with q elements.

The Twisted Grassmann Graph

Discovered by Van Dam & Koolen (2005).
Same parameters as $J_q(2k+1,k)$.

Take a 1-space P of $V(5,q)$.

Vertices: 2-spaces of $V(5,q)$ not on P, 4-spaces on P.

Adjacency of x and y:

- As in $J_q(5,2)$ if x, y are not on P.
- If x, y on P, then x, y are adjacent.
- Otherwise, x, y are adjacent iff x, y incident.

Twisted Grassmann Graphs II

$V(n, q) =$ vector space of dimension n over field with q elements.

The Twisted Grassmann Graph (again)

This description is due to Munemasa (2017).

Take a polarity σ in the residue of a point P of $V(5, q)$.

Vertices: 2-spaces of $V(5, q)$.

Adjacency of x and y:

- As in $J_q(5, 2)$ if x, y are not on P.
- As in $J_q(4, 2)$ if x, y are on P.
- If $P \nsubseteq x, P \subseteq y$, then x, y are adjacent iff $x \subseteq y^\sigma$.

Diagrams

The **moral reason** for non-isomorphy:
$V(2k, q)$ has a polarity which does not extend to $V(2k+1, q)$.

We can check all the buildings for such polarities!

(Remind the speaker to draw some rank ≥ 3 diagrams.)

Conclusion 1: We are left with the diagrams A_n, D_5, E_6, E_7.

Conclusion 2: For DRGs, only A_n and D_5 remain.

The $D_{5,5}(q)$ Graph

Put

$$Q(x) = x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 + x_9x_{10}.$$

Then the restriction of $V(10, q)$ to vectors with $Q(x) = 0$ yields the geometry $O^+(10, q)$.

The $D_{5,5}(q)$ Graph

Vertices: 5-spaces of one type of $O^+(10, q)$.

Adjacency: meet in a 3-space.

Up to now: Only known family with parameters

$$v = (q+1)(q^2+1)(q^3+1)(q^4+1), \qquad k = q(q^2+1)\frac{q^5-1}{q-1},$$

$$\lambda = q-1+q^2(q+1)(q^2+q+1), \qquad \mu = (q^2+1)(q^2+q+1).$$

The Twisted $D_{5,5}(q)$ Graph

Put

$$Q(x) = x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 + x_9x_{10}.$$

Then the restriction of $V(10, q)$ to vectors with $Q(x) = 0$ yields the geometry $O^+(10, q)$.

The Twisted $D_{5,5}(q)$ Graph

Discovered by I. (2023). Same parameters as $D_{5,5}(q)$.

Let P be a 1-space of $O^+(10, q)$.

Vertices: 5-spaces of one type not on P, 2-space through P.

Adjacency of x and y:

- As in $D_{5,5}(q)$ if x, y are 5-spaces.
- Being coplanar in $D_{5,5}$ if x, y are 2-spaces.
- Otherwise, x, y are adjacent iff $x \cap y$ is a 1-space.

The Twisted $D_{5,5}(q)$ Graph

Put

$$Q(x) = x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8 + x_9x_{10}.$$

Then the restriction of $V(10, q)$ to vectors with $Q(x) = 0$ yields the geometry $O^+(10, q)$.

The Twisted $D_{5,5}(q)$ Graph (again)

Discovered by I. (2023). Same parameters as $D_{5,5}(q)$.

Identify the residue of some 1-space P of $O^+(10, q)$ with $O^+(8, q)$.

σ: polarity of $O^+(8, q)$ interchanging one type of 4-spaces and 1-spaces.

Vertices: 5-spaces of one type.

Adjacency of x and y:

- As in $D_{5,5}(q)$ if x, y are not on P.
- As in $D_{5,5}(q)$ if x, y are both on P.
- Otherwise, x, y are adjacent iff $x \cap y^\sigma$ is a 3-space.

Maximal Cliques and the Automorphism Group

Well-known: Two types of maximal cliques of $D_{5,5}(q)$.

Easy Exercise: Seven types of maximal cliques of the twisted $D_{5,5}(q)$.

Proof: See what happens to $D_{5,5}(q)$.

Corollary

We have $Aut(twisted\ D_{5,5}(q)) = Stab(Aut(D_{5,5}(q)), P)$.

Thank you for your attention!