Pappus Configurations in Finite Planes

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Overview

1. Affine planes
2. Desargues and Pappus configurations
3. Conjectures
4. An analytic approach

Definitions

Finite sets of points and lines

We will consider a set of elements, $\mathcal{P}$, that we call *points* and another set of elements, $\mathcal{L}$, that we call *lines*, such that lines are subsets of the point set.

On and Incident

For a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$ such that $p \in \ell$, we say that the point p is *on* the line ℓ or the line ℓ is *on* the point p. We also say that the point p and the line ℓ are *incident*.

Concurrent and collinear

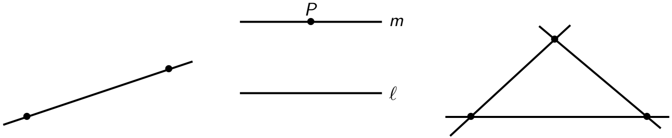
To indicate that a point is an element of each line in a set of lines, we say that the lines from the set are *concurrent* at the point. To indicate that each point in a set of points is an element of the same line, we say that the points in the set are *collinear*.

Affine planes

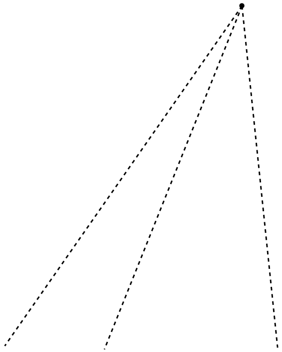
The concept of an affine plane reminds us of high school geometry. It is a generalization of the Euclidean plane with only the notion of incidence.

Affine plane axioms

- There exists a unique line on two distinct points.
- For a line ℓ and a point P not on ℓ, there exists a unique line m on P such m and ℓ have no common point.
- There exist three noncollinear points.



The Desargues configuration



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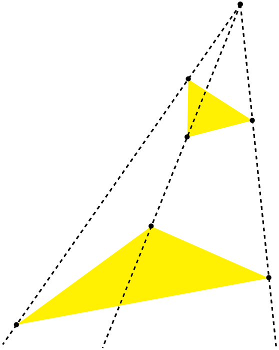

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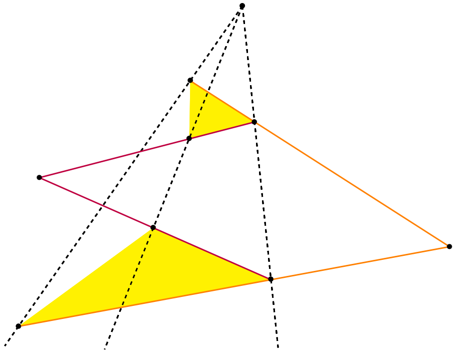

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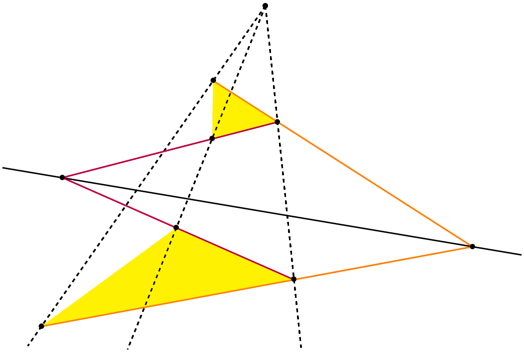

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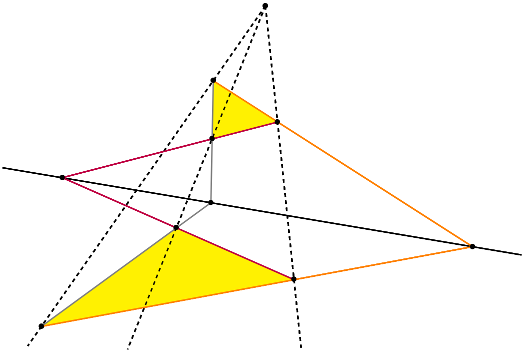
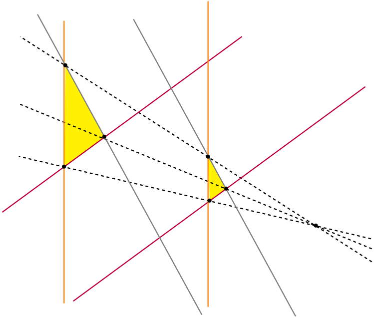
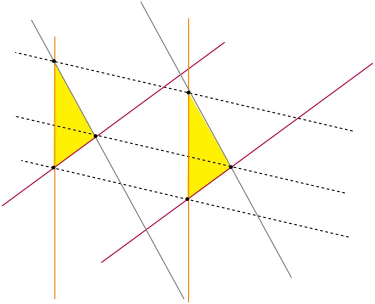


Particular cases of the Desargues configuration





The Desargues theorem

Theorem

For every triple of concurrent lines and any pair of triangles with vertices on those lines, the lines through the pairs of corresponding sides of the triangles intersect in three collinear points.

- The Desargues theorem does not hold in every affine plane. (Hilbert [1899], Moulton [1902])
- Roughly speaking, if the Desargues theorem holds in a plane, the coordinatizing algebraic system is guaranteed to have associative multiplication.

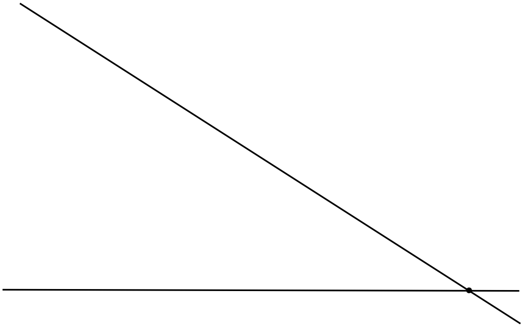
The Desargues configuration

Suppose the Desargues theorem does not hold in a plane.

Can we still find at least one Desargues configuration?

- In infinite planes, no. (Hall, Jr. [1943])
- The Desargues configuration exists in every finite affine plane with more than 10 points. (Ostrom [1957])

The Pappus configuration



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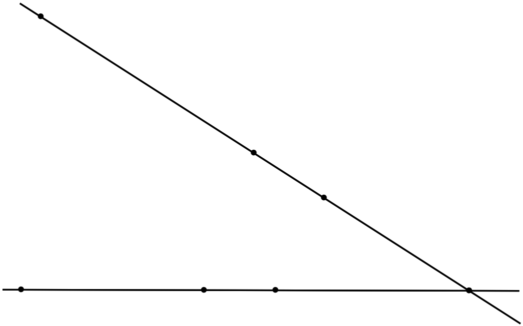

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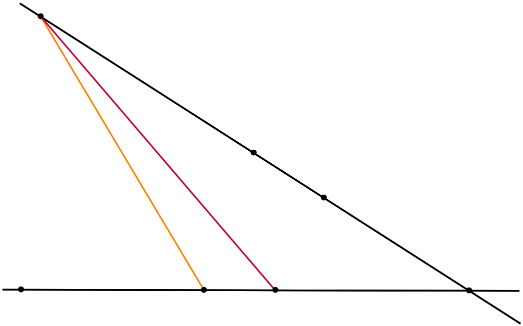

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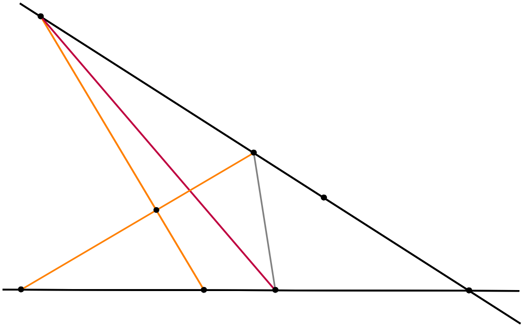

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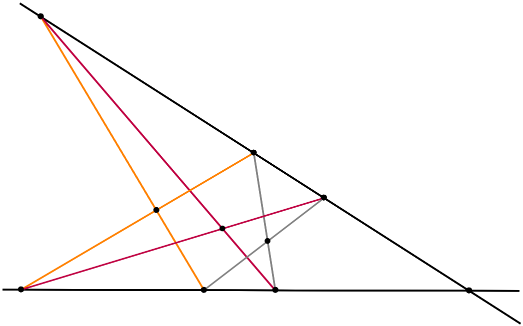

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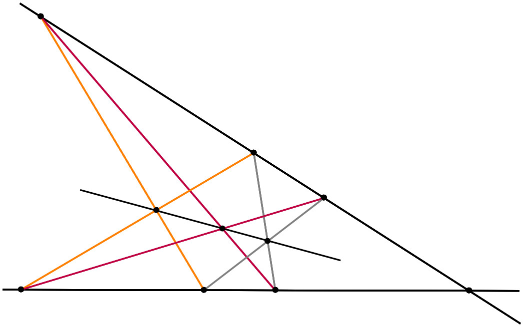
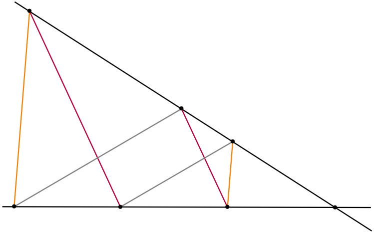
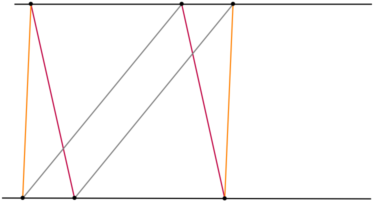


Particular cases of the Pappus configuration





The Pappus theorem

Theorem

For every pair of lines and every triple of points on each line, A_1, B_1, C_1 and A_2, B_2, C_2, respectively, the intersection points of A_1B_2 and B_1A_2, A_1C_2 and C_1A_2, B_1C_2 and C_1B_2 are collinear.

- The Pappus theorem does not hold in every affine plane. (Hilbert [1899], Moulton [1902])
- Roughly speaking, if the Pappus theorem holds in a plane, the coordinatizing algebraic system is guaranteed to to have commutative multiplication.

The Pappus configuration

Suppose the Pappus theorem does not hold in a plane.

Can we still find at least one Pappus configuration?

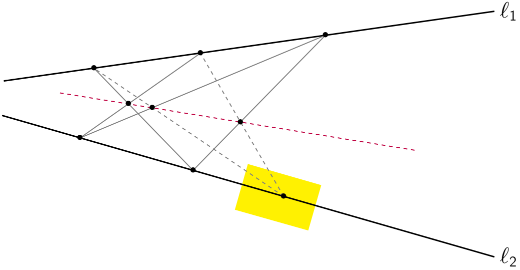
- In infinite planes, no. (Hall, Jr. [1943])
- In finite planes: This is unknown.

Motivated by Ostrom's result [1957] which shows that the Desargues configuration exists on every triple of lines in all finite planes, we attempt to determine the situation for the Pappus configuration.

Conjectures

The 3+2-conjecture

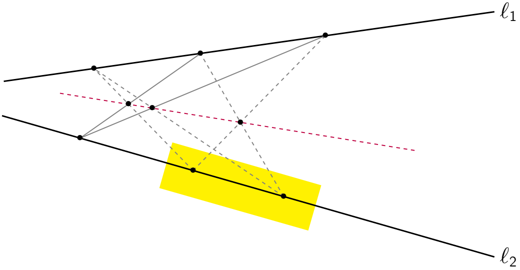
In finite Hall affine planes, for every pair of lines ℓ_1, ℓ_2, every triple of points on ℓ_1 and every pair of points on ℓ_2, one more point can be found on line ℓ_2 that defines a Pappus configuration.



Conjectures

The 3+1-conjecture

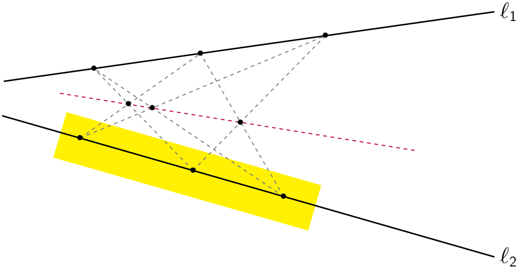
In finite Hall affine planes, for every pair of lines ℓ_1, ℓ_2, every triple of points on ℓ_1 and every point on ℓ_2, two more points can be found on line ℓ_2 that define a Pappus configuration.



Conjectures

The 3+0-conjecture

In finite Hall affine planes, for every pair of lines ℓ_1, ℓ_2, and every triple of points on ℓ_1, three more points can be found on line ℓ_2 that define a Pappus configuration.



A search for Pappus configurations in finite planes: Strong versions of existence conjectures

The 3+2-conjecture and 3+1-conjecture were tested in Magma. Some planes were constructed using built-in commands, some were constructed algebraically, and some were downloaded from Eric Moorhouse's website: http://ericmoorhouse.org/.

A selection of planes for which the 3+2-conjecture failed:

- Hall planes: orders $9, 16, 25, 49$
- Hughes planes: orders $9, 25, 49$
- Dickson Near-field planes: order 49
- Zassenhaus Near-field planes: order 49

A selection of planes for which the 3+1-conjecture passed:

- Hall planes: orders $16, 25$
- Hughes planes: order 25
- Czerwinski & Oakden planes: order 25: $a1, a6, b3, b6$
- Rao planes: order 25: $a5, a7$

An analytic approach

In the infinite Euclidean affine plane we have the usual Cartesian coordinates. Points are represented as ordered pairs of real numbers. Lines are described by linear equations.

It was a deep question asked by Hilbert [1899]:

Having an affine plane (defined as an incidence system satisfying the three axioms), can one "attach" an algebraic object $\{\mathcal{S}, +, \times\}$, so that points correspond to ordered pairs of elements of $\mathcal{S}$ and lines are described by linear equations?

Related work

Veblen-Wedderburn systems [1907]

Veblen, a geometer, and Wedderburn, an algebraist,

- constructed finite analytic models $\{\mathcal{S}, +, \times\}$,
- introduced an algebraic system now called a Veblen-Wedderburn system, and
- constructed a finite affine plane with 81 points that cannot be coordinatized by a field.

Hall planes [1943]

Hall, Jr., the eponym for Hall planes,

- generalized the work of Veblen and Wedderburn, and
- built the planes with his construction of an infinite class of Veblen-Wedderburn systems.

Algebraic structure of Hall affine planes

(Right) Hall system: base field $\mathbb{F} = GF(q)$, $q = p^n$, prime p.

The elements of the Hall system are $\mathbb{H} = \{(a, b) \mid a, b \in \mathbb{F}\}$. Choose a polynomial $f(x) = x^2 - rx - s$, irreducible over $\mathbb{F}$. In terms of addition and multiplication from $\mathbb{F}$, the operations in the Hall system $\mathbb{H}$ are:

A. $(a, b) + (c, d) = (a + c, b + d)$

M1. $(a, b) \cdot (c, 0) = (ac, bc)$

M2. $(a, b) \cdot (c, d) = (ac - bc^2d^{-1} + bcd^{-1}r + bd^{-1}s, ad - bc + br)$, for $d \neq 0$

- The multiplication is neither commutative nor associative. It is right distributive but not left distributive over addition.

Sample collineations

The collineation group $\mathfrak{G}$ (Hughes [1959])

of Hall planes can be generated from six non-distinct categories of collineations. We identify $a = (a, 0)$ and $\mathbf{y} = (y_1, y_2)$ for $a, y_1, y_2 \in \mathbb{F}$ and $\mathbf{y} \in \mathbb{H}$.

The translation group $\mathfrak{T}$: for $\tau \in \mathfrak{T}$, $\tau = \tau(\mathbf{a}, \mathbf{b})$, $\mathbf{a}, \mathbf{b} \in \mathbb{H}$:

- $\mathbf{y} = \mathbf{x}\mathbf{m} + \mathbf{k} \mapsto \mathbf{y} = \mathbf{x}\mathbf{m} + (\mathbf{k} + \mathbf{m}\mathbf{a} + \mathbf{b})$
- $\mathbf{x} = \mathbf{k} \mapsto \mathbf{x} = \mathbf{k} + \mathbf{a}$

The linear group $\mathfrak{L}$: for $\zeta \in \mathfrak{L}$, $\zeta = \zeta(a, b)$, $a, b \in \mathbb{F}$:

- $\mathbf{y} = \mathbf{x}(r + ba^{-1}) + \mathbf{k} \mapsto \mathbf{x} = \mathbf{k}a$, if $a \neq 0$
- $\mathbf{x} = \mathbf{k} \mapsto \mathbf{y} = \mathbf{x}(-ba^{-1}) - \mathbf{k}a^{-1}(b^2 + rab - sa^2)$, if $a \neq 0$

Based on the entire group of collineations, only four pairs of lines need to be checked for a complete proof. This is determined by transcribing Hughes' work to apply to right VW-systems in Hall affine planes.

Sample of a symbolic proof

$$\{\alpha,\ \beta,\ \alpha\,m,\ \beta\,m\}\ \text{join}\ \{0,\ 0,\ t,\ v\} \longrightarrow \left\{\frac{t\,(v+(-m+r)\,\beta)+\alpha\,(m^2\,\beta-s\,\beta-m\,(v+r\,\beta))}{-v\,\alpha+t\,\beta},\ \frac{v^2+(-2\,m+r)\,v\,\beta+(m^2-m\,r-s)\,\beta^2}{-v\,\alpha+t\,\beta},\ t,\ v\right\}$$

$$\{\gamma,\ \delta,\ \gamma\,m,\ \delta\,m\}\ \text{join}\ \{0,\ 0,\ \eta,\ \theta\} \longrightarrow \left\{m-\frac{\theta}{\delta},\ 0,\ 0,\ \theta\right\}$$

$$\left\{\frac{t\,(v+(-m+r)\,\beta)+\alpha\,(m^2\,\beta-s\,\beta-m\,(v+r\,\beta))}{-v\,\alpha+t\,\beta},\ \frac{v^2+(-2\,m+r)\,v\,\beta+(m^2-m\,r-s)\,\beta^2}{-v\,\alpha+t\,\beta},\ t,\ v\right\}\ \text{meet}\ \left\{m-\frac{\theta}{\delta},\ 0,\ 0,\ \theta\right\} \longrightarrow$$

$$\Big\{\frac{(v\,\alpha-t\,\beta)\,(v-\theta)}{v^2+(-2\,m+r)\,v\,\beta+(m^2-m\,r-s)\,\beta^2},\ \Big(\delta\,\big(v\,\alpha\,(2\,m\,\delta-r\,\delta-2\,\theta)\,(v-\theta)+t\,\big(-v^2\,\delta+2\,v\,\beta\,\theta+\beta\,\big(-m^2\,\beta\,\delta+s\,\beta\,\delta+m\,\delta\,(r\,\beta+2\,\theta)-\theta\,(r\,\delta+2\,\theta)\big)\big)\big)\Big)\Big/$$
$$\Big((v\,\alpha-t\,\beta)\,\big(m^2\,\delta^2-s\,\delta^2+\theta\,(r\,\delta+\theta)-m\,\delta\,(r\,\delta+2\,\theta)\big)\Big),\ \frac{(v\,\alpha-t\,\beta)\,(v-\theta)\,(m\,\delta-\theta)}{(v^2+(-2\,m+r)\,v\,\beta+(m^2-m\,r-s)\,\beta^2)\,\delta},$$
$$\Big(-m^3\,t\,\beta^2\,\delta^2+m^2\,\delta\,\big(2\,v^2\,\alpha\,\delta-v\,\alpha\,\delta\,\theta+t\,\beta\,(r\,\beta\,\delta+(\beta+\delta)\,\theta)\big)+\theta\,\big(v\,\alpha\,\big(r\,v\,\delta-s\,\delta^2+2\,v\,\theta-\theta^2\big)+t\,\big(v^2\,\delta-2\,v\,\beta\,\theta+\beta\,\big(s\,\delta\,(-\beta+\delta)+\theta^2\big)\big)\big)-$$
$$m\,\delta\,\big(v\,\alpha\,\big(r\,v\,\delta+4\,v\,\theta-2\,\theta^2\big)+t\,\big(v^2\,\delta-2\,v\,\beta\,\theta+\beta\,\big(-s\,\beta\,\delta+r\,\beta\,\theta+2\,\theta^2\big)\big)\big)\Big)\Big/\Big((v\,\alpha-t\,\beta)\,\big(m^2\,\delta^2-s\,\delta^2+\theta\,(r\,\delta+\theta)-m\,\delta\,(r\,\delta+2\,\theta)\big)\Big)\Big\}$$

Figure: Difficulties introduced with coordinates

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Thank you!