On the isomorphism of certain Q-polynomial association schemes

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(joint work with Alessandro Siciliano)

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Association Schemes

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$X =$ finite set, $|X| \geq 2$

$d =$ positive integer

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Definition

(X, R) is a *$d-$class association scheme* if :

A1. R is a partition of $X \times X$ with $R_0 = \{(x, x) | x \in X\}$;

A2. $R_i^{-1} = \{(y, x) | (x, y) \in R_i\} = R_i$, $i = 0, ..., d$;

A3. for each $(x, y) \in R_k$,

$$p_{i,j}^{(k)} = |\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}| = p_{j,i}^{(k)}$$

does not depend on (x, y).

Definition

Two schemes $(X, \{R_i\}_{0\leq i\leq d})$ and $(X', \{R_i'\}_{0\leq i\leq d})$ are *isomorphic* if there exists a bijection φ from X to X' and a permutation σ of $\{1,\ldots,d\}$ such that

$$(x,y)\in R_i \iff (\varphi(x),\varphi(y))\in R'_{\sigma(i)}.$$

The Bose–Mesner Algebra

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$\mathbb{R}(X, X)$ = the set of all the $|X|$-matrices over $\mathbb{R}$

Definition

$A_i \in \mathbb{R}(X, X)$ with

$$A_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

is called the *adjacency matrix* of R_i.

Theorem (Bose-Mesner, 1952)

Let (X, R) be an association scheme with d classes. Then

$$\mathcal{A} = \langle A_0, ..., A_d \rangle_{\mathbb{R}}$$

is a commutative subalgebra in $\mathbb{R}(X, X)$ such that:

i. $\dim \mathcal{A} = d + 1$;

ii. $D = D^T$, for each $D \in \mathcal{A}$.

$\mathcal{A}$ is the so-called *Bose-Mesner algebra* of (X, R).

Corollary

i. $\mathcal{A}$ *admits* $d+1$ *common maximal eigen-spaces* $V_0, \ldots, V_d$, *where* $V_0 = \langle \mathbf{1} \rangle$, *such that*

$$\mathbb{R}^{|X|} = V_0 \perp \ldots \perp V_d.$$

Corollary

i. $\mathcal{A}$ *admits* $d+1$ *common maximal eigen-spaces* $V_0, \ldots, V_d$*, where* $V_0 = \langle \boldsymbol{1} \rangle$*, such that*

$$\mathbb{R}^{|X|} = V_0 \perp \ldots \perp V_d.$$

ii. $\mathcal{A}$ *admits a unique basis of minimal idempotent matrices* $\{E_0, \ldots, E_d\}$*.*

The Eigenmatrices

Definition

The matrices P and Q such that

$$(A_0\ A_1\ \ldots\ A_d) = (E_0\ E_1\ \ldots\ E_d)P$$

and

$$(E_0\ E_1\ \ldots\ E_d) = |X|^{-1}(A_0\ A_1\ \ldots\ A_d)Q$$

are the *first* and the *second eigenmatrix* of (X, R), respectively.

Definition

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A scheme is *Q-polynomial* if, after a reordering of the eigenspaces, there are polynomials q_i of degree i such that $E_i = q_i(E_1)$, where multiplication is done entrywise.

The Hollmann-Xiang association scheme

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Let $\mathcal{C}$ be a *non-degenerate conic* in $\mathrm{PG}(2, q^2)$:

$$\mathcal{C} = \{\langle(1, t, t^2)\rangle : t \in \mathbb{F}_{q^2}\} \cup \{\langle(0, 0, 1)\rangle\}$$

A line ℓ of $\mathrm{PG}(2, q^2)$ is called a *passant* if $|\ell \cap \mathcal{C}| = 0$.

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Let $\overline{\mathcal{C}}$ be the *extension* of $\mathcal{C}$ in $\mathrm{PG}(2, q^4)$.

An *elliptic* line of $\overline{\mathcal{C}}$ is the extension $\overline{\ell}$ of a passant ℓ of $\mathcal{C}$.

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Then

$$\overline{\ell} \cap \overline{\mathcal{C}} = \{\langle(1, t, t^2)\rangle, \langle(1, t^{q^2}, t^{2q^2})\rangle\},$$

for some $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$.

$\mathcal{E}$ = the set of all the elliptic lines of $\overline{\mathcal{C}}$

$\mathcal{X}$ = the set of all pairs $\mathbf{t} = \{t, t^{q^2}\}$ with t in $\mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$

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The identification

$$\begin{array}{rccc} \xi : & \mathbb{F}_{q^4} \cup \{\infty\} & \longleftrightarrow & \overline{\mathcal{C}} \\ & t & \longleftrightarrow & \langle (1, t, t^2) \rangle \\ & \infty & \longleftrightarrow & \langle (0, 0, 1) \rangle \end{array}$$

induces the bijection

$$\begin{array}{ccc} \mathcal{X} & \longleftrightarrow & \mathcal{E} \\ \mathbf{t} = \{t, t^{q^2}\} & \longleftrightarrow & \ell_{\mathbf{t}}, \end{array}$$

where $\ell_{\mathbf{t}} = \overline{\ell}$ with $\overline{\ell} \cap \overline{\mathcal{C}} = \{\langle (1, t, t^2) \rangle, \langle (1, t^{q^2}, t^{2q^2}) \rangle\}$.

q even

For any two distinct pairs $\mathbf{s} = \{s, s^{q^2}\}, \mathbf{t} = \{t, t^{q^2}\} \in \mathcal{X}$, let

$$\rho(s,t) = \frac{(s+t)(s^{q^2}+t^{q^2})}{(s+t^{q^2})(s^{q^2}+t)} \in \mathbb{F}_{q^2} \setminus \{0,1\}$$

Note that $\rho(s,t)$ is the *cross-ratio* of (s, s^{q^2}, t, t^{q^2}).

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Note that $\rho(s,t)$ is the *cross-ratio* of (s, s^{q^2}, t, t^{q^2}).

From the properties of the cross-ratio it is possible to define the *cross-ratio* of $\{\mathbf{s}, \mathbf{t}\}$ as the pair

$$\{\rho(s,t), \rho(s,t)^{-1}\}.$$

Theorem (Hollmann-Xiang, 2006)

Under the identification ξ, the action of $\mathrm{PGL}(2, q^2)$ on $\mathcal{E} \times \mathcal{E}$ gives rise to an association scheme on $\mathcal{X}$ with $q^2/2 - 1$ classes $R_{\{\lambda,\lambda^{-1}\}}$, $\lambda \in \mathbb{F}_{q^2} \setminus \{0, 1\}$, where

$$(\mathbf{s}, \mathbf{t}) \in R_{\{\lambda,\lambda^{-1}\}} \iff \{\rho(s, t), \rho(s, t)^{-1}\} = \{\lambda, \lambda^{-1}\}.$$

The fusion scheme

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$\mathbf{T}_0(q^r)$ = the set of all the elements of $\mathbb{F}_{q^r}$ with *absolute trace* zero

$$\mathbf{T}_0 = \mathbf{T}_0(q^2); \qquad \mathbf{S}_0^* = \mathbf{T}_0(q) \setminus \{0\}; \qquad \mathbf{S}_1 = \mathbb{F}_q \setminus \mathbf{S}_0.$$

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For any two distinct pairs $\mathbf{s}, \mathbf{t} \in \mathcal{X}$, define

$$\widehat{\rho}(\mathbf{s}, \mathbf{t}) = \frac{1}{\rho(s,t) + \rho(s,t)^{-1}}$$

Since

$$\widehat{\rho}(\mathbf{s}, \mathbf{t}) = \left(\frac{1}{\rho(s,t)+1}\right)^2 + \left(\frac{1}{\rho(s,t)+1}\right),$$

then

$$\operatorname{Im}\widehat{\rho} \subset \mathbf{T}_0.$$

Theorem (Hollmann-Xiang, 2006)

The following relations are defined on $\mathcal{X}$:

R_1: $(\mathbf{s}, \mathbf{t}) \in R_1$ if and only $\widehat{\rho}(s,t) \in \mathbf{S}_0^*$;

R_2: $(\mathbf{s}, \mathbf{t}) \in R_2$ if and only $\widehat{\rho}(s,t) \in \mathbf{S}_1$;

R_3: $(\mathbf{s}, \mathbf{t}) \in R_3$ if and only $\widehat{\rho}(s,t) \in \mathbf{T}_0 \setminus \mathbb{F}_q$.

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R_3: $(\mathbf{s}, \mathbf{t}) \in R_3$ if and only $\widehat{\rho}(s,t) \in \mathbf{T}_0 \setminus \mathbb{F}_q$.

Then $(\mathcal{X}, \{R_i\}_{i=0}^3)$ is a 3-class association scheme which is a *fusion* of the previous scheme.

The Penttila-Williford association schemes

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Assume q even, and let

$H(3,q^2)$ be the *unitary polar space of rank 2* of $\mathrm{PG}(3,q^2)$;

$W(3,q)$ be a *symplectic polar space of rank 2* embedded in $H(3,q^2)$;

$Q^-(3,q)$ be an *orthogonal polar space of rank 1* embedded in $W(3,q)$.

For any line l of $H(3, q^2)$ disjoint from $W(3, q)$, let $\mathcal{S}_l$ denote the set of the (extended) lines of $W(3, q)$ that meet l.

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Definition

A *relative hemisystem of $H(3, q^2)$ with respect to $W(3, q)$* is a set $\mathcal{H}$ of lines of $H(3, q^2)$ disjoint from $W(3, q)$ such that every point of $H(3, q^2)$ not in $W(3, q)$ lies on exactly $q/2$ lines of $\mathcal{H}$.

Theorem (Penttila-Williford, 2011)

Let $\mathcal{H}$ be a relative hemisystem of $H(3, q^2)$ with respect to $W(3, q)$. Then a Q-polynomial (not P-polynomial) 3-class association scheme is constructed on $\mathcal{H}$ through the following relations:

Theorem (Penttila-Williford, 2011)

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$\widetilde{R}_1$: $(l, m) \in \widetilde{R}_1$ if and only $|l \cap m| = 1$;

$\widetilde{R}_2$: $(l, m) \in \widetilde{R}_2$ if and only $l \cap m = \emptyset$ and $|\mathcal{S}_l \cap \mathcal{S}_m| = 1$;

$\widetilde{R}_3$: $(l, m) \in \widetilde{R}_3$ if and only $l \cap m = \emptyset$ are $|\mathcal{S}_l \cap \mathcal{S}_m| = q + 1$.

The existence of relative hemisystems

$\mathrm{PO}^-(4, q)$ = the *stabilizer* of $Q^-(3, q)$ in $\mathrm{PGU}(4, q^2)$

$\mathrm{P\Omega}^-(4, q)$ = the *commutator* subgroup of $\mathrm{PO}^-(4, q)$

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$$\mathrm{P}\Omega^-(4,q) = \text{the } \textit{commutator} \text{ subgroup of } \mathrm{PO}^-(4,q)$$

Theorem (Penttila-Williford, 2011)

$\mathrm{P}\Omega^-(4,q)$ has two orbits on the lines of $H(3,q^2)$ disjoint from $W(3,q)$, both of them relative hemisystems with respect to $W(3,q)$.

Tanaka (private communication to Penttila and Williford)

The 3-class association schemes found by Hollmann and Xiang have the same parameters as the 3-class schemes derived from the Penttila-Williford relative hemisystems.

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Question:

Are the above 3-class association schemes isomorphic?

The key-stone:

$(\mathrm{PSL}(2,q^2),\mathrm{PG}(1,q^2))$ *and* $(\mathrm{P}\Omega^-(4,q),Q^-(3,q))$ *are permutationally isomorphic for all prime powers* q.

A non-standard geometric setting

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From now on q is even.

$$\widehat{V} = \{(\alpha, x^q, x, \beta) : \alpha, \beta \in \mathbb{F}_q, x \in \mathbb{F}_{q^2}\} \hookrightarrow V(4, q^2)$$

$W(\widehat{V})$ = the symplectic polar space arising from the intersection of $H(3, q^2)$ with $\mathrm{PG}(\widehat{V})$

$$\widehat{\mathcal{Q}} = \{\langle(1, t^q, t, t^{q+1})\rangle : t \in \mathbb{F}_{q^2}\} \cup \{\langle(0, 0, 0, 1)\rangle\}$$

is a $Q^-(3, q)$ of $W(\widehat{V})$

Let

$$\begin{array}{rccc} \theta: & \mathrm{PG}(1,q^2) & \longrightarrow & \widehat{\mathcal{Q}} \\ & \langle(1,t)\rangle & \mapsto & \langle(1,t^q,t,t^{q+1})\rangle \\ & \langle(0,1)\rangle & \mapsto & \langle(0,0,0,1)\rangle \end{array}$$

and

$$\begin{array}{rccc} \chi: & \mathrm{PSL}(2,q^2) & \longrightarrow & \mathrm{P}\Omega^-(\widehat{V}) \\ & g & \mapsto & g\otimes g^q\ , \end{array}$$

where $\otimes$ is the *Kronecher product*.

Proposition

$(\mathrm{PSL}(2,q^2), \mathrm{PG}(1,q^2))$ and $(\mathrm{P}\Omega^-(\widehat{V}), \widehat{\mathcal{Q}})$ are permutationally isomorphic (for all prime powers q), i.e.

$$\begin{array}{ccccc}
\mathrm{PSL}(2,q^2) & \times & \mathrm{PG}(1,q^2) & \longrightarrow & \mathrm{PG}(1,q^2) \\
\chi\downarrow & & \theta\downarrow & & \theta\downarrow \\
\mathrm{P}\Omega^-(\widehat{V}) & \times & \widehat{\mathcal{Q}} & \longrightarrow & \widehat{\mathcal{Q}}
\end{array}$$

is a commutative diagram.

The Pentilla-Williford relative hemisystem

For any $t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}$, let

$$\theta(t) = \langle (1, t^q, t, t^{q+1}) \rangle, \quad \theta(t^{q^2}) = \langle (1, t^{q^3}, t^{q^2}, t^{q^3+q^2}) \rangle$$

and $M_{\mathbf{t}} = \langle \theta(t), \theta(t^{q^2}) \rangle$. Note that $M_{\mathbf{t}}$ is a line of $\mathrm{PG}(3, q^4)$.

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and $M_{\mathbf{t}} = \langle \theta(t), \theta(t^{q^2}) \rangle$. Note that $M_{\mathbf{t}}$ is a line of $\mathrm{PG}(3, q^4)$.

Lemma

i. For each $\mathbf{t} = \{t, t^{q^2}\}$, $m_{\mathbf{t}} = M_{\mathbf{t}} \cap \mathrm{PG}(3, q^2)$ is a line of $H(3, q^2)$, which is disjoint from $W(\widehat{V})$.

ii. $\{m_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$ is one of the Penttila-Williford relative hemisystem.

Betting...

Betting...

Finding a bijection between the sets

$$\mathcal{X} = \{\mathbf{t} = \{t, t^{q^2}\} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$$

and

$$\mathcal{H} = \{m_{\mathbf{t}} : t \in \mathbb{F}_{q^4} \setminus \mathbb{F}_{q^2}\}$$

such that the relations respectively defined on them, after a proper reordering, are preserved.

Lemma

The map

$$\begin{array}{rccc} \varphi : & \mathcal{X} & \to & \mathcal{H} \\ & \mathbf{t} & \mapsto & m_{\mathbf{t}} \end{array}$$

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is a bijection.

Is φ the winning bijection?

A dual setting

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The Klein correspondence κ (q even)

$$\text{lines of } \mathrm{PG}(3,q^2) \quad \longleftrightarrow \quad \text{points of } Q^+(5,q^2)$$

$$\text{lines of } H(3,q^2) \quad \longleftrightarrow \quad \text{points of } Q^-(5,q)$$

$$\text{lines of } W(3,q) \quad \longleftrightarrow \quad \text{points of } Q(4,q).$$

The Klein correspondence κ (q even)

$$\text{lines of } \; H(3,q^2) \quad \longleftrightarrow \quad \text{points of } \; Q^-(5,q)$$

$$\text{lines of } \; W(\widehat{V}) \quad \longleftrightarrow \quad \text{points of (which?) } \; Q(4,q).$$

Another non-standard geometric setting

$$\widetilde{V} = \{(x, x^q, y, y^q, z, z^q) : x, y, z \in \mathbb{F}_{q^2}\} \hookrightarrow V(6, q^2)$$

$\widetilde{\mathcal{Q}}$: $xz^q + x^q z + y^{q+1} = 0$ is a $Q^-(5, q)$ in $\mathrm{PG}(\widetilde{V})$

$$\Gamma = \{\langle(x, x^q, c, c, z, z^q)\rangle : x, z \in \mathbb{F}_{q^2}, c \in \mathbb{F}_q\}$$

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$$\Gamma \;=\; \{\langle (x, x^q, c, c, z, z^q) \rangle : x, z \in \mathbb{F}_{q^2}, c \in \mathbb{F}_q\}$$

Then

$$Q(4, q) = \Gamma \cap \widetilde{\mathcal{Q}} = \kappa(W(\widehat{V}))$$

$$m_{\mathbf{t}} \in \mathcal{H} \quad \stackrel{\kappa}{\longleftrightarrow} \quad P_{\mathbf{t}} \in Q^{-}(5,q) \setminus Q(4,q)$$

$$m_{\mathbf{t}} \in \mathcal{H} \quad \overset{\kappa}{\longleftrightarrow} \quad P_{\mathbf{t}} \in Q^{-}(5,q) \setminus Q(4,q)$$

$$\mathcal{S}_{m_{\mathbf{t}}} \quad \overset{\kappa}{\longleftrightarrow} \quad \widetilde{\mathcal{O}}_{\mathbf{t}} = Q(4,q) \cap P_{\mathbf{t}}^{\perp}$$

Looking at some *special* planes of $\mathrm{PG}(\widetilde{V})$

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For $\mathbf{s} \neq \mathbf{t}$, let

$$\Pi_{\mathbf{s},\mathbf{t}} = \langle \Gamma^{\perp}, P_{\mathbf{s}}, P_{\mathbf{t}} \rangle,$$

and $\widetilde{Q}_{\mathbf{s},\mathbf{t}}$ be the restriction of $\widetilde{Q}$ on $\Pi_{\mathbf{s},\mathbf{t}}$.

Looking at some *special* planes of $\mathrm{PG}(\widetilde{V})$

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Then

$$\mathrm{Rad}(\Pi_{\mathbf{s},\mathbf{t}}) = \langle v_{\mathbf{s},\mathbf{t}} \rangle.$$

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and $\widetilde{Q}_{\mathbf{s},\mathbf{t}}$ be the restriction of $\widetilde{Q}$ on $\Pi_{\mathbf{s},\mathbf{t}}$.

Then

$$\mathrm{Rad}(\Pi_{\mathbf{s},\mathbf{t}}) = \langle v_{\mathbf{s},\mathbf{t}} \rangle.$$

Two cases are possible:

$$\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) = 0$$

$$\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) \neq 0$$

First case: $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) = 0$

$\Pi_{\mathbf{s},\mathbf{t}} \cap Q^-(5,q)$ consists of two distinct lines through $\langle v_{\mathbf{s},\mathbf{t}} \rangle$.

First case: $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) = 0$

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Two sub-cases:

i. $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are collinear in $Q^-(5,q)$

ii. $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are NOT collinear in $Q^-(5,q)$

Subcase *i.* : $P_{\mathbf{s}}$ *and* $P_{\mathbf{t}}$ *are collinear in* $Q^{-}(5,q)$

$P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are collinear in $Q^{-}(5,q)$ if and only if

$m_{\mathbf{s}} = \kappa^{-1}(P_{\mathbf{s}})$ and $m_{\mathbf{t}} = \kappa^{-1}(P_{\mathbf{t}})$ are concurrent in $H(3,q^2)$,

that is $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_1$.

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that is $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_1$.

On the other hand, $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are collinear in $Q^{-}(5,q)$ if and only if

$$\frac{1}{\rho(s,t)+1} = \frac{(s+t^{q^2})(s^{q^2}+t)}{(s^{q^2}+s)(t^{q^2}+t)} \in \mathbb{F}_q,$$

if and only if $\widehat{\rho}(\mathbf{s},\mathbf{t}) \in \mathbf{S}_0^*$, that is $(\mathbf{s},\mathbf{t}) \in R_1$.

Subcase *ii.* : $P_{\mathbf{s}}$ *and* $P_{\mathbf{t}}$ *are NOT collinear in* $Q^-(5,q)$

$P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are NOT collinear in $Q^-(5,q)$ if and only if

$m_{\mathbf{s}} = \kappa^{-1}(P_{\mathbf{s}})$ and $m_{\mathbf{t}} = \kappa^{-1}(P_{\mathbf{t}})$ are NOT concurrent in $H(3,q^2)$,

that is $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_2$.

Subcase *ii.* : $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are NOT collinear in $Q^{-}(5,q)$

$P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are NOT collinear in $Q^{-}(5,q)$ if and only if $m_{\mathbf{s}} = \kappa^{-1}(P_{\mathbf{s}})$ and $m_{\mathbf{t}} = \kappa^{-1}(P_{\mathbf{t}})$ are NOT concurrent in $H(3,q^2)$, that is $(m_{\mathbf{s}}, m_{\mathbf{t}}) \in \widetilde{R}_2$.

On the other hand, $P_{\mathbf{s}}$ and $P_{\mathbf{t}}$ are NOT collinear in $Q^{-}(5,q)$ if and only if

$$\left(\frac{1}{\rho(s,t)+1}\right)^q + \frac{1}{\rho(s,t)+1} = 1,$$

that is $\widehat{\rho}(\mathbf{s},\mathbf{t}) \in \mathbf{S}_1$, i.e. $(\mathbf{s},\mathbf{t}) \in R_2$.

Second case: $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) \neq 0$

$\Pi_{\mathbf{s},\mathbf{t}} \cap Q^{-}(5,q)$ is a non-degenerate conic with nucleus $\langle v_{\mathbf{s},\mathbf{t}} \rangle$.

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$\Pi_{\mathbf{s},\mathbf{t}} \cap Q^{-}(5,q)$ is a non-degenerate conic with nucleus $\langle v_{\mathbf{s},\mathbf{t}} \rangle$.

Then $|\widetilde{\mathcal{O}}_{\mathbf{t}} \cap \widetilde{\mathcal{O}}_{\mathbf{s}}| = q+1$ if and only if

$\mathcal{S}_{m_{\mathbf{t}}} = \kappa^{-1}(\widetilde{\mathcal{O}}_{\mathbf{t}})$ and $\mathcal{S}_{m_{\mathbf{s}}} = \kappa^{-1}(\widetilde{\mathcal{O}}_{\mathbf{s}})$ meet in $q+1$ lines of $W(\widehat{V})$,

that is $(m_{\mathbf{t}}, m_{\mathbf{s}}) \in \widetilde{R}_3$.

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that is $(m_{\mathbf{t}}, m_{\mathbf{s}}) \in \widetilde{R}_3$.

On the other hand, $\widetilde{Q}_{\mathbf{s},\mathbf{t}}(v_{\mathbf{s},\mathbf{t}}) \neq 0$ if and only if $(\mathbf{s}, \mathbf{t}) \in R_3$ by exclusion.

Summing up...

The bijection

$$\begin{array}{rccc} \varphi : & \mathcal{X} & \rightarrow & \mathcal{H} \\ & \mathbf{t} & \mapsto & m_{\mathbf{t}} \end{array}$$

enjoys the property

$$(\mathbf{s}, \mathbf{t}) \in R_i \iff (m_{\mathbf{s}}, m_{\mathbf{t}}) = \varphi(\mathbf{s}, \mathbf{t}) \in \widetilde{R}_i, \qquad i = 1, 2, 3,$$

i.e. ...

Theorem G.Monzillo - A. Siciliano

The Hollmann-Xiang and Pentilla-Williford Q-polynomial (but not P-polynomial) association schemes are isomorphic.