Projective Metrics

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$\leftarrow$ more specific/structured | more general $\rightarrow$

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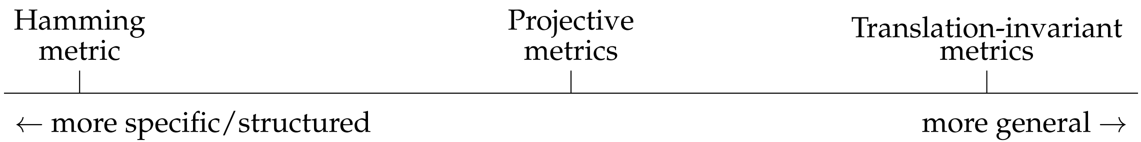
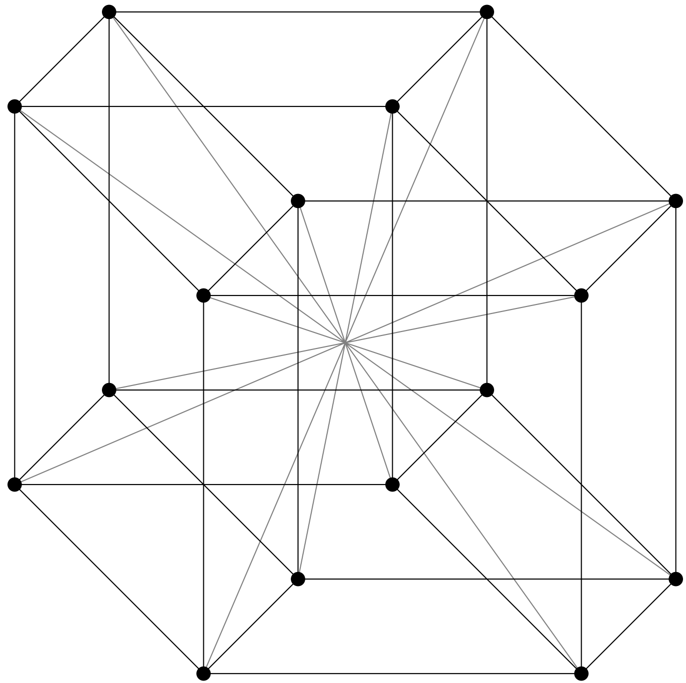
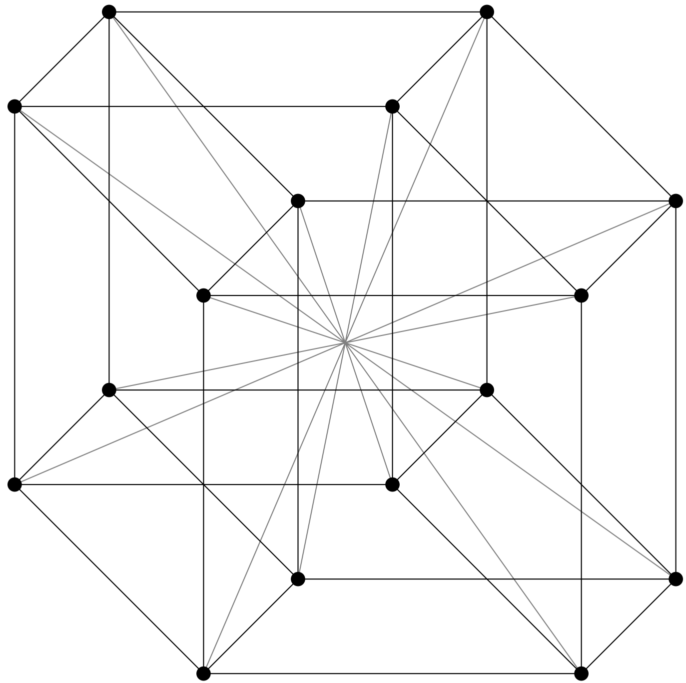
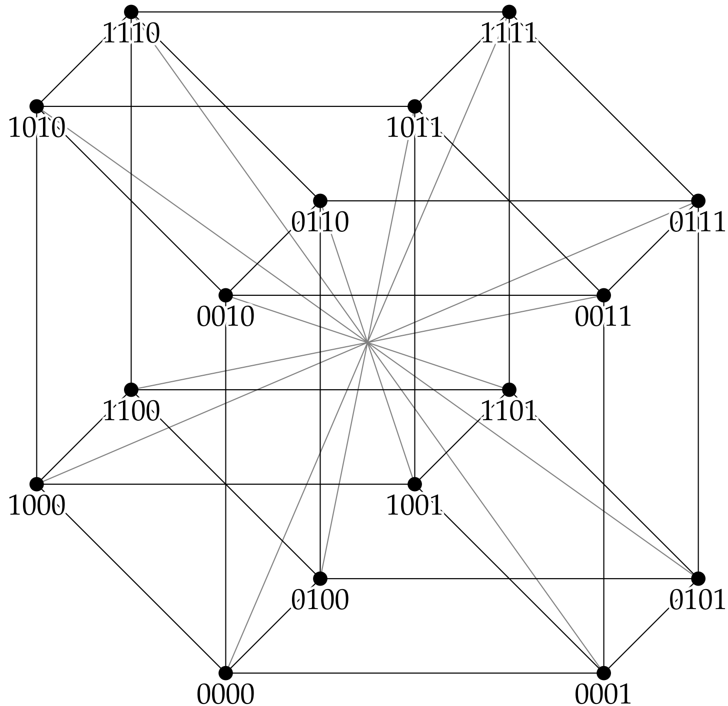




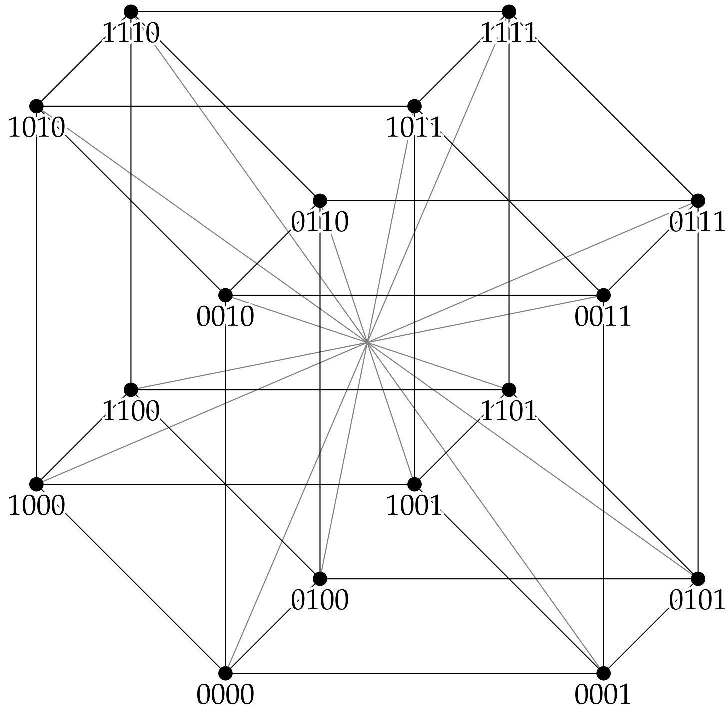
Stongly regular Clebsch graph / Greenwood–Gleason graph



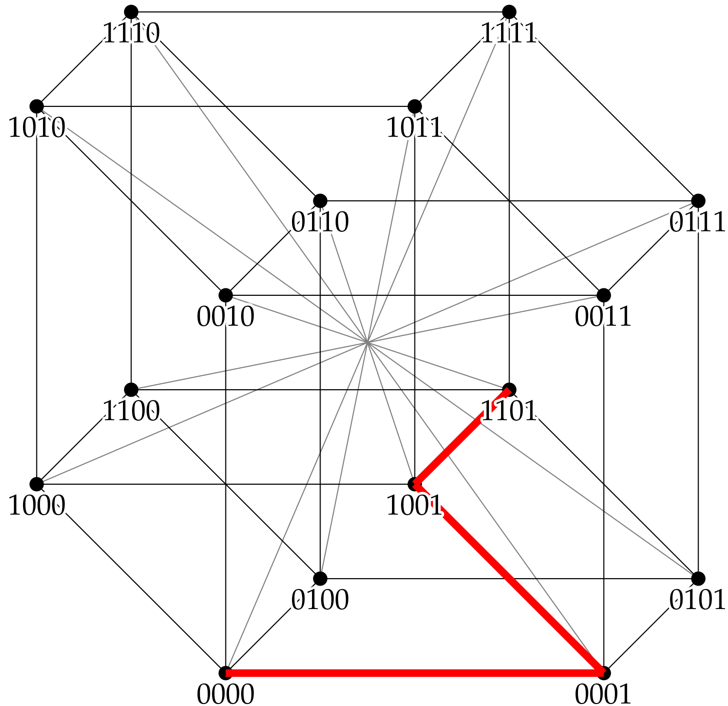
Vertices: vectors of $\mathbb{F}_2^4$



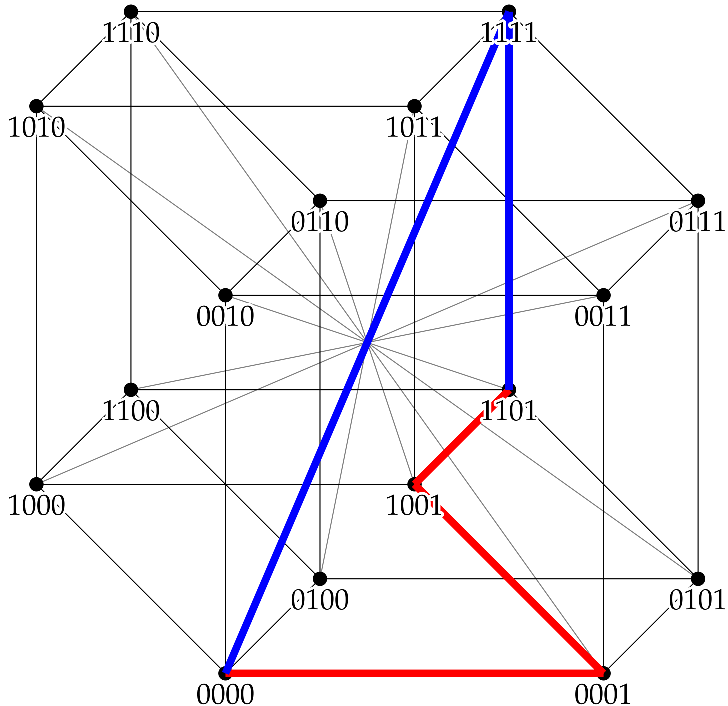
Distance from 0000 to 1101:



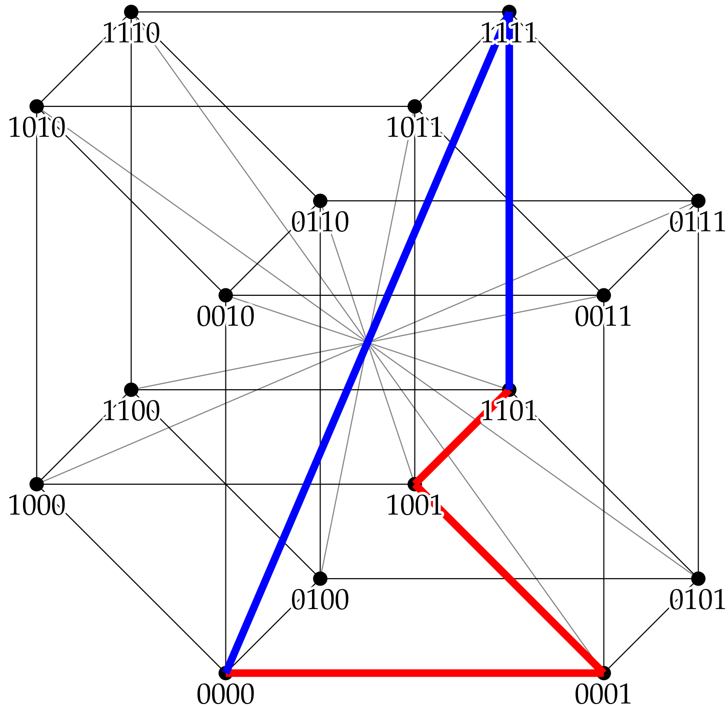
Distance from 0000 to 1101: red: 3,



Distance from 0000 to 1101: red: 3, blue: 2

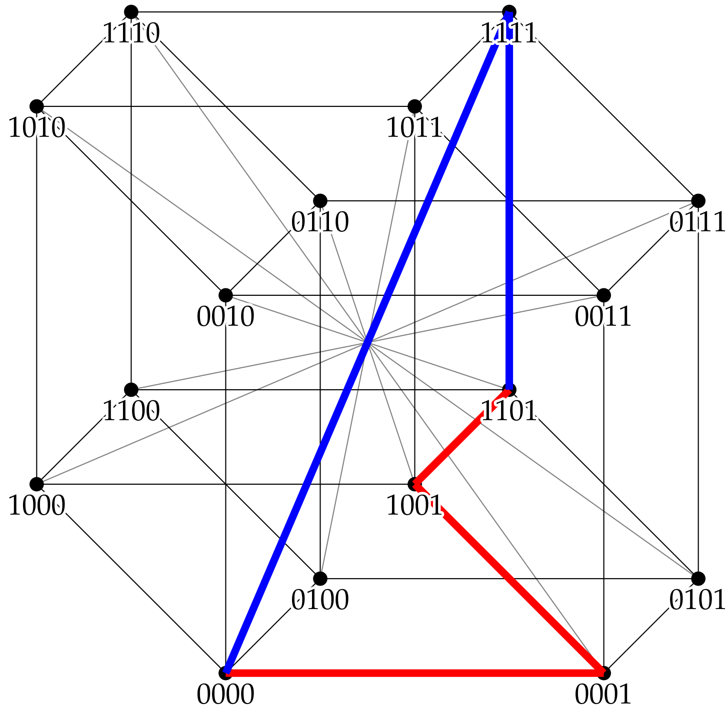


Graph distance on Clebsch graph $=$ **Phase-rotation metric/distance** on $\mathbb{F}_2^4$



Graph distance on Clebsch graph = **Phase-rotation metric/distance** on $\mathbb{F}_2^4$

An edge is a Hamming error or the all-bits-flip error



Hamming metric

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \rightarrow \quad \mathrm{wt}_{\mathrm{Hamming}} = 3$$

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$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \mathrm{wt}_{\mathrm{Phase\text{-}Rot}} = 1 + 1 = 2$$

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Phase-rotation metric

More: burst metric, tensor metric, combinatorical metrics, etc.

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Translation invariant metric/distance function $d(\cdot, \cdot)$ on V satisfies

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A translation invariant metric is **projective** iff for every $x \in V$:

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The set of 1-dim subspaces (projective points)

$$\mathcal{F} = \{\, \langle f_i \rangle \;\mid f_i \in V,\, \mathrm{wt}(f_i) = 1\}$$

is called the **spanning family**.

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The **projective metric** $d_{\mathcal{F}}(\cdot,\cdot) : V \times V \to \mathbb{N}_{\geq 0}$ corresponding to $\mathcal{F}$ is

$$d_{\mathcal{F}}(x, y) := \mathrm{wt}_{\mathcal{F}}(y - x).$$

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Q: For fixed t, how many v have $\mathrm{wt}_{\mathcal{F}}(v) = t$?

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  Flats spanned by points in $\mathcal{F}$; $\mathrm{wt}_{\mathcal{F}}(v)$ is smallest rank of such a flat that contains v.
- **Matroids:**
  - $\mathrm{wt}_{\mathcal{F}}(v)$ is cardinality of smallest subset of $\mathcal{F}$ whose closure contains v.
  - View $\mathcal{F}$ as ground set of representable matroid, study dependent and independent sets.

  Q: General ways to calculate $\mathrm{wt}_{\mathcal{F}}(v)$?
  Q: For fixed t, how many v have $\mathrm{wt}_{\mathcal{F}}(v) = t$?

- Please let me know if you know a (partial) answer in any of these contexts! :)

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Theorem (General Singleton-type bound)

(S. 202?) Let $\mathcal{C} \subseteq V$ be a subset and let $d = \min\{d_{\mathcal{F}}(x, y) \mid x \neq y \in \mathcal{C})\}$. Then

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Coincides with Singleton bounds for specific projective metrics!

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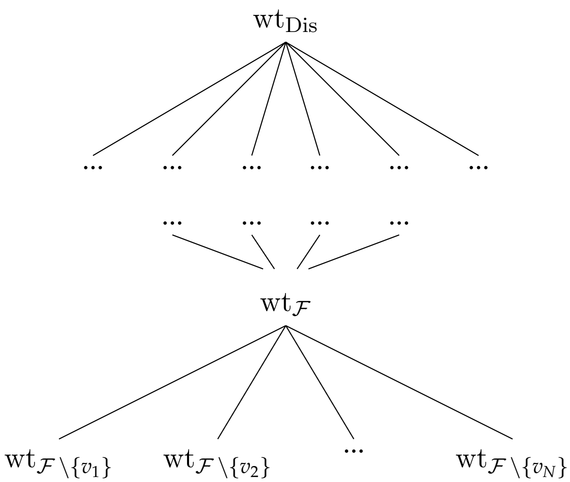

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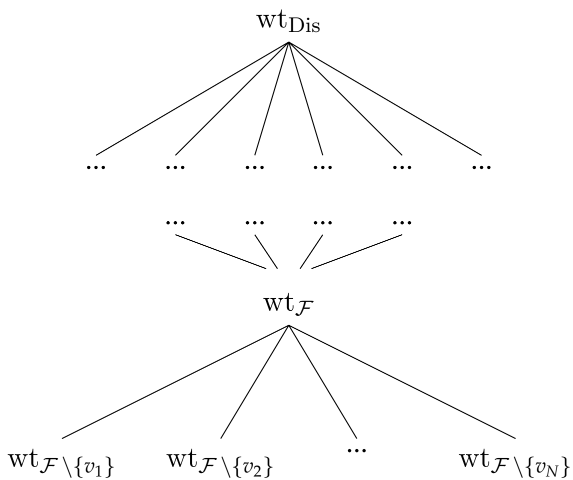

Ideas on how this might work are very welcome! :)

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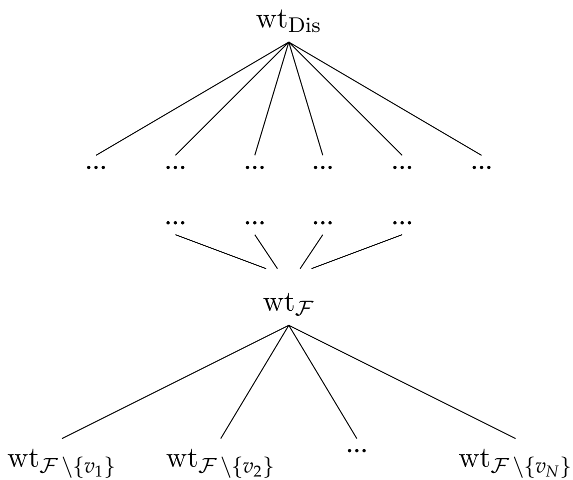

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Let me know if you know more projective metrics!