Polar Geometry and (Belgian) Friends

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Finite Geometry & Friends

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Finite classical polar spaces

Definitions

Let $\mathcal{P}$ be a finite classical polar space. Hence $\mathcal{P}$ is a member of one of the following classes: a symplectic space $W(2n+1, q)$, a parabolic quadric $Q(2n, q)$, an hyperbolic quadric $Q^+(2n+1, q)$, an elliptic quadric $Q^-(2n+1, q)$ or an Hermitian variety $H(n, q)$ (q a square). A projective subspace of maximal dimension contained in $\mathcal{P}$ is called a *generator* of $\mathcal{P}$. The vector dimension of a generator of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$. $\mathcal{P}_{d,e}$ will denote a polar space of rank $d \geq 2$ as follows:

$\mathcal{P}_{d,e}$	$Q^+(2d-1, q)$	$H(2d-1, q)$	$W(2d-1, q)$	$Q(2d, q)$	$H(2d, q)$	$Q^-(2d+1, q)$
e	0	1/2	1	1	3/2	2

$\mathcal{M}_{\mathcal{P}_{d,e}}$ will denote the set of generators of the polar space $\mathcal{P}_{d,e}$, while $\mathcal{M}_{\mathcal{P}_{d-1,e}}$ will denote the set of generators passing through a fixed point.

Finite classical polar spaces

Known facts

Let $\theta_d := \frac{q^{d+1}-1}{q-1} = 1 + q + \ldots + q^d$.

Proposition

1. *$\mathcal{P}_{d,e}$ has $|\mathcal{P}_{d,e}| = \theta_d(q^{d+e-1} + 1)$ points;*
2. *the number of generators is $|\mathcal{M}_{\mathcal{P}_{d,e}}| = \prod_{i=1}^{d}(q^{d+e-i} + 1)$;*
3. *each generator contains θ_d points;*
4. *through each point there pass $|\mathcal{M}_{\mathcal{P}_{d-1,e}}| = \prod_{i=2}^{d}(q^{d+e-i} + 1)$ generators.*

Finite classical polar spaces

Research problems

Nowadays, some research problems related to finite classical polar space are:

- existence of spreads and ovoids;
- existence of regular systems and *m*-ovoids;
- upper or lower bounds on partial spreads and partial ovoids.

Moreover, polar spaces are in relation with combinatorial objects as regular graphs, block designs and association schemes.

Partial ovoids and m-ovoids

Definition

1. *An ovoid $\mathcal{O}$ of a polar space $\mathcal{P}_{d,e}$ is a set of points of $\mathcal{P}_{d,e}$ such that every generator contains exactly one point of $\mathcal{O}$.*
2. *A partial ovoid $\mathcal{O}$ of a polar space $\mathcal{P}_{d,e}$ is a set of points of $\mathcal{P}_{d,e}$ such that every generator contains at most one point of $\mathcal{O}$. A partial ovoid is said to be maximal if it is maximal with respect to set-theoretic inclusion.*
3. *An m-ovoid $\mathcal{O}$ of a polar space $\mathcal{P}_{d,e}$ is a set of points of $\mathcal{P}_{d,e}$ such that every generator contains exactly m point of $\mathcal{O}$, $0 \leq m \leq \theta_d$.*

Partial ovoids and m-ovoids

Partial ovoids of $W(3, q)$, q odd

	Parital ovoid in $W(3, q)$	Partial spread in $Q(4, q)$
Size $\mathbf{q+1}$	$q+1$ points on a non-isotropic line	Regulus of $q+1$ lines in $Q^+(3, q)$

Theorem (G. Tallini, 1988)

If q even, $W(3, q)$ has an ovoid of size q^2+1.
If q odd, $W(3, q)$ has no ovoids. Moreover, a maximal partial ovoid has size at most q^2-q+1.

Partial ovoids and m-ovoids

Partial ovoids of $W(3, q)$, q odd

$\mathcal{O}(\mathcal{P}_{d,e})$:= size of the largest maximal partial ovoid of $\mathcal{P}_{d,e}$

Theorem (M. Ceria, J. De Beule, F. Pavese, V.S., 2022)

If $q = p^{2n}$, $p \neq 2, 3$,

$$\frac{q^{\frac{3}{2}} + 3q - q^{\frac{1}{2}} + 3}{3} < \mathcal{O}(W(3, q)) \leq q^2 - q + 1.$$

Partial ovoids and m-ovoids

The new construction

1. $\mathcal{C} = \{(1, -3t, t^2, t^3) | t \in F_q\} \cup \{(0, 0, 0, 1)\}$ *twisted cubic*
2. $G \simeq PGL(2, q)$ group of projectivities fixing $\mathcal{C}$
3. G stabilizes $W(3, q)$ given by $x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1$
4. $K \leq G$, $K \simeq PGL(2, \sqrt{q})$, fixes a twisted cubic $\bar{\mathcal{C}} \subset \mathcal{C}$ of a $PG(3, \sqrt{q}) \subset PG(3, q)$.
5. ℓ line of $PG(3, q)$, $|\ell \cap PG(3, \sqrt{q})| = \sqrt{q} + 1$
6. $|\ell \cap (\mathcal{C} \setminus \bar{\mathcal{C}})| = 2$, if $q \equiv -1 \pmod 3$, $|\ell \cap \bar{\mathcal{C}}| = 2$, if $q \equiv 1 \pmod 3$

Partial ovoids and m-ovoids

The new construction

Proposition

$\exists P \in \ell \setminus PG(3, \sqrt{q})$, such that:

- *P^K is a partial ovoid of $W(3, q)$*
- *P^K has size $\frac{q^{\frac{3}{2}} - q^{\frac{1}{2}}}{3}$*
- *$P^K \cup \mathcal{C}$ is a partial ovoid of $W(3, q)$ of size $\frac{q^{\frac{3}{2}} + 3q - q^{\frac{1}{2}} + 3}{3}$*

$P^K \cup \mathcal{C}$ is not maximal!

Partial ovoids and m-ovoids

Non-existance results for m-ovoids

$$\mathcal{P}'_{d,e} \in \{Q^-(2d+1, q), W(2d-1, q), H(2d, q^2)\}$$

Theorem (J. Bamberg, S. Kelly, M. Law, T. Penttila, 2007)

Consider an m-ovoid $\mathcal{O}$ in the polar space $\mathcal{P}'_{d,e}$. Then $m \geq b$, with b given in the table below.

$\mathcal{P}'_{d,e}$	b
$Q^-(2d+1, q)$	$\dfrac{-3+\sqrt{9+4q^{d+1}}}{2(q-1)}$
$W(2d-1, q)$	$\dfrac{-3+\sqrt{9+4q^{d}}}{2(q-1)}$
$H(2d, q^2)$	$\dfrac{-3+\sqrt{9+4q^{2d+1}}}{2(q^2-1)}$

Partial ovoids and *m*-ovoids

Characterstic function

Lemma

Suppose that $\mathcal{O}$ is a set of points in $\mathcal{P}'_{d,e}$ with ambient projective space $PG(n, q)$. Then $\mathcal{O}$ is an m-ovoid if and only if for every point $p \in PG(n, q)$

$$|p^{\perp} \cap \mathcal{O}| = \begin{cases} (m-1)(q^{d+e-2}+1)+1, & p \in \mathcal{O}, \\ m(q^{d+e-2}+1), & p \in PG(n,q) \setminus \mathcal{O}\,. \end{cases}$$

Let $\mathcal{O}$ be an *m*-ovoid with characteristic vector χ, and let π be any subspace of the ambient projective space. The *weight* of π is then defined as $\mu(\pi) = \sum_{P \in \pi} \chi_P$, i.e. the number of points of $\mathcal{O}$ contained in π.

$$\mu(p^{\perp}) + q^{d+e-2}\mu(p) = m(q^{d+e-2}+1)$$

Partial ovoids and *m*-ovoids

Weighted *m*-ovoids

Definition

Consider $\mu : \mathcal{P}_{d,e} \to \mathbb{N}$ *such that for every subspace* π *of* $PG(n, q)$ *it holds that* $\mu(\pi) = \sum_{p\in\pi} \mu(p)$. *Then we call* μ *a weighted m-ovoid of* $\mathcal{P}_{d,e}$ *if for every point p it holds that*

$$\mu(p^\perp) + q^{d+e-2}\mu(p) = m(q^{d+e-2} + 1).$$

Lemma

Suppose that μ *is a weighted m-ovoid in* $\mathcal{P}'_{d,e}$, *then for every j-dimensional space* π *in* $PG(n, q)$,

$$\mu(\pi^\perp) + q^{d+e-j-2}\mu(\pi) = m(q^{d+e-j-2} + 1).$$

Partial ovoids and m-ovoids

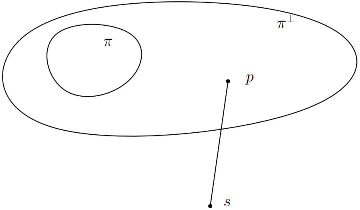
Technical lemmas

Lemma

Suppose that μ is a weighted m-ovoid in $\mathcal{P}'_{d,e}$ and π is a j-subspace, $0 \leq j \leq d-1$. If $\mu(\pi^\perp \setminus \pi) \neq 0$, then

$$m(q^{d+e-j-3}+1)(m(q^{d+e-1}+1)-\mu(\pi))+q^{d+e-2}\sum_{p\in\pi^\perp\setminus\pi}\mu(p)^2 =$$

$$= m(q^{d+e-2}+1)(m-\mu(\pi))(q^{d+e-j-2}+1)+q^{d+e-j-3}\sum_{p\in\mathcal{P}'_{d,e}\setminus\pi}\mu(p)\mu(\langle p,\pi\rangle)+\sum_{s\not\in\pi^\perp}\mu(s^\perp\cap\pi)\,. \quad (1)$$



Partial ovoids and m-ovoids

Technical lemmas

Corollary

Suppose that μ is a weighted m-ovoid in $\mathcal{P}'_{d,e}$ and p_0 is a point such that $\mu(p_0) < m$. Then

$$m(q^{d+e-3}+1)(m(q^{d+e-1}+1)-\mu(p_0))+q^{d+e-2}\sum_{p\in p_0^{\perp}\setminus\{p_0\}}\mu(p)^2=$$

$$=m(q^{d+e-2}+1)^2(m-\mu(p_0))+q^{d+e-3}\sum_{p\in\mathcal{P}'_{d,e}\setminus\{p_0\}}\mu(p)\mu(\langle p_0,p\rangle)$$

Lemma

Let $\mathcal{O}$ be a non-trivial m-ovoid of $\mathcal{P}'_{d,e}$, with $m \geq 2$. then

$$(q-1)^2m^2+3(q-1)m-q^{d+e-1}-q-2\geq 0.$$

Partial ovoids and m-ovoids

Improvement for Theorem 17

Theorem (J. De Beule, J. Mannaert, V. S., 2023)

Consider a non-trivial m-ovoid $\mathcal{O}$ in $\mathcal{P}'_{d,e}$, $d \geq 2$, $d > 2$ for $W(2d-1, q)$. Then $m \geq b$, with b given in the table below.

$\mathcal{P}'_{d,e}$	b
$Q^-(2d+1, q)$	$\dfrac{-3+\sqrt{9+4(q^{d+1}+q-2)}}{2(q-1)}$
$W(2d-1, q)$	$\dfrac{-3+\sqrt{9+4(q^{d}+q-2)}}{2(q-1)}$
$H(2d, q^2)$	$\dfrac{-3+\sqrt{9+4(q^{2d+1}+q^2-2)}}{2(q^2-1)}$

Partial ovoids and m-ovoids
Main Theorem

Theorem

Assume that $\mathcal{O}$ is an m-ovoid in $\mathcal{P}'_{d,e}$ and π is $(d-2)$-subspace such that $\mu(\pi^\perp \setminus \{\pi\}) \neq 0$, with $\mu(\pi) = \mu$ then

$$m^2(q^{d+e-1} - q^{d+e-2} - q^{2e-1} - q^e) + m\left[q^e\left(\mu(q^{d-2} + 2q^{e-1} + q) + q^{d-2} + q^{e-1}\right)\right]$$

$$-\mu\left(q^{d+2e-2} + q^{d+e-2} + (1+\mu)(q^{2e-1} + q^{e-1}) + q^{d+2e-1}\frac{q^{d-2}-1}{q-1}\right) \geq 0$$

Partial ovoids and m-ovoids

Main Theorem

Theorem (J. De Beule, J. Mannaert, V. S., 2023)

Let $q > 2$ and $d \geq 3$. Suppose that $\mathcal{O}$ is an m-ovoid in $\mathcal{P}'_{d,e}$, with $d \geq 4$ **OR** *$e \in \{1, \frac{3}{2}\}$ and $(d, q, e) \neq (3, 3, 1)$. Then it holds that*

$$m \geq \frac{-d(1 + \frac{2}{q^{d-e-1}}) + \sqrt{d^2(1 + \frac{2}{q^{d-1}})^2 + 4(q-2)(d-1)(q^{e+1}\frac{q^{d-2}-1}{q-1} + q^e + 1)}}{2(q-1)}.$$

This bound asymptotically converges to

$$m \geq \frac{-d + \sqrt{d^2 + 4(d-1)(q-2)q^{d+e-2}}}{2(q-1)}.$$

Partial ovoids and m-ovoids

Main Theorem

Theorem (J. De Beule, J. Mannaert, V. S., 2023)

Suppose that $\mathcal{O}$ is an m-ovoid in $Q^-(7, q)$, for $q > 2$, then

$$m \geq \frac{-9 + \sqrt{9(1 + \frac{2}{q^2})^2 + 8\left(q - \frac{7}{3}\right)(q^3 + q^2 + 1)}}{2(q-1)}.$$

Partial ovoids and m-ovoids

Summary tables

Bounds for m-ovoids of $W(2d-1,3)$

d	Bound from Theorem 22	Bound from Theorem 24
4	$m \geq 4$	$m \geq 5$
5	$m \geq 8$	$m \geq 10$
6	$m \geq 13$	$m \geq 20$
7	$m \geq 23$	$m \geq 39$
100	$m \geq 3,59 \cdot 10^{23}$	$m \geq 2,53 \cdot 10^{24}$

Partial ovoids and m-ovoids

Summary tables

Bounds for m-ovoids of $Q^-(2d+1, 3)$

d	Bound from Theorem 22	Bound from Theorem 24
4	$m \geq 8$	$m \geq 8$
5	$m \geq 13$	$m \geq 18$
6	$m \geq 23$	$m \geq 36$
7	$m \geq 40$	$m \geq 69$
100	$m \geq 6,22 \cdot 10^{23}$	$m \geq 4,37 \cdot 10^{24}$

Partial ovoids and m-ovoids
Summary tables

Bounds for m-ovoids of $H(2d, 9)$

d	Bound from Theorem 22	Bound from Theorem 24
3	$m \geq 6$	$m \geq 8$
4	$m \geq 18$	$m \geq 29$
5	$m \geq 53$	$m \geq 99$
6	$m \geq 158$	$m \geq 330$
7	$m \geq 474$	$m \geq 1085$
100	$m \geq 1,12 \cdot 10^{47}$	$m \geq 1,04 \cdot 10^{48}$

Partial ovoids and m-ovoids
Summary tables

Bounds for m-ovoids of $Q^-(7, q)$

q	Bound from Theorem 22	Bound from Theorem 25
3	$m \geq 4$	$m \geq 2$
4	$m \geq 5$	$m \geq 5$
5	$m \geq 6$	$m \geq 6$
7	$m \geq 8$	$m \geq 10$
8	$m \geq 9$	$m \geq 11$
$3^5 = 243$	$m \geq 244$	$m \geq 345$

Graph $NU(3, q^2)$ and block graphs

Strongly regular graphs

$$G := (V(G), E(G))$$

$V = V(G)$ is a non-empty set, of element called *vertices*
$E = E(G)$ is the set of *edges*, together with an *incidence function* $\phi : E \to V \times V$. If $\phi(e) = \{u, v\}$ we say that e *joins* u and v, and those are called *adjacent vertices* or *neighbours*.

Definition

A strongly regular graph with parameters (v, k, λ, μ) *is a graph with* v *vertices, each vertex lies on* k *edges, any two adjacent vertices have* λ *common neighbours and any two non-adjacent vertices have* μ *common neighbours.*

Graph $NU(3, q^2)$ and block graphs

Let consider the projective space $PG(n, q^2)$, together with a non-degenerate Hermitian variety $H = H(n, q^2)$.
Let $n \geq 2$ and $\varepsilon = (-1)^{n+1}$.

Definition

$NU(n+1, q^2)$ is the graph whose vertex set is $PG(n, q^2) \setminus H$, and two vertices are adjacent if they lie on the same tangent line.

Graph $NU(3, q^2)$ and block graphs

Parameters of $NU(3, q^2)$

Proposition

$NU(n+1, q^2)$ is a strongly regular graph with parameters:

$$v = \frac{q^n(q^{n+1} - \varepsilon)}{q+1}$$

$$k = (q^n + \varepsilon)(q^{n-1} - \varepsilon)$$

$$\lambda = q^{2n-3}(q+1) - \varepsilon q^{n-1}(q-1) - 2$$

$$\mu = q^{n-2}(q+1)(q^{n-1} - \varepsilon).$$

Corollary

$NU(3, q^2)$ has parameters

$$(q^4 - q^3 + q^2, (q^2 - 1)(q+1), 2(q^2 - 1), (q+1)^2).$$

Graph $NU(3, q^2)$ and block graphs

Automorphism group of $Aut(NU(3, q^2))$

Theorem (F. Romaniello, V. S., 2022)

Let $G_2 = Aut(NU(3, q^2))$ be the automorphism group of the graph $NU(3, q^2)$:

1. *if $q \neq 2$, $G_2 \cong P\Gamma U(3, q)$, the semilinear collineation group stabilizing the Hermitian curve $H(2, q^2)$;*
2. *if $q = 2$, $G_2 \cong S_3 \wr S_4 \cong S_3^4 \rtimes S_4$.*

Graph $NU(3, q^2)$ and block graphs

The dual *block graph*

Definition

An unital $\mathcal{U}$ *is a* $2-(a^3+1, a+1, 1)$ *block design,* $a \geq 3$*, i.e. a set of* a^3+1 *points arranged into blocks of size* $a+1$*, such that each pair of distinct points is contained in exactly one block.*

Definition

Let $\mathcal{U}$ *be an unital. The block graph of* $\mathcal{U}$ *is the graph whose vertices are the blocks of the design, and two distinct blocks define adjacent vertices if they share a point.*

Graph $NU(3,q^2)$ and block graphs

Maximal cliques of block graphs

Work in progress with S. Adriaensen, J. De Beule, F. Romaniello.

Conjecture

$Aut(\mathcal{U}) \cong Aut(\Gamma_{\mathcal{U}})$.

D. Mezőfi, G. P. Nagy, *Algorithms and libraries of abstract unitals and their embeddings, Version 0.5 (2018)*, (GAP package), https://github.com/nagygp/UnitalSZ

1. *Classical unital*: $Aut(H(2,q^2)) \cong P\Gamma U(3,q)$;
2. *Ree unital*: $Aut(ReeU(3)) \cong Ree(3) \cong P\Gamma L(2,8)$;
3. *Buekenhout-Metz orthogonal unital*, $q=3$: $Aut(BM(3)) \cong (C_3 \times C_3 \times C_3) \rtimes Q_8$;
4. *Buekenhout-Tits unital*, $q=3$: $Aut(BT(3)) \cong ((C_4 \times C_4) \rtimes C_8) \rtimes C_6$;
5. *Bagchi-Bagchi unital*, $n=6$: $Aut(BB(6)) \cong C_7 \rtimes (C_{31} \rtimes C_{30})$.

Graph $NU(3, q^2)$ and block graphs

Maximal cliques of block graphs and block graphs

Theorem (**Hoffman's clique bound**)

The size of a maximal clique of a k-regular graph $\mathcal{G}$ is bounded by

$$\omega(\mathcal{G}) \leq 1 + \frac{k}{|\lambda|},$$

where λ is the smallest eigenvalue.

Corollary

The size of a maximal clique of $\mathcal{G} = NU(3, q^2)$ is bounded by

$$\omega(\mathcal{G}) \leq 1 + \frac{(q^2 - 1)(q + 1)}{q + 1} = q^2.$$

Graph $NU(3, q^2)$ and block graphs

Maximal cliques of block graphs

Theorem (M. De Boeck, 2015)

Let $\mathcal{U}$ be a unital of order q and let S be a maximal Erdős–Ko–Rado set on $\mathcal{U}$.

1. *If $q \geq 5$ then either $|S| = q^2$ and S is a point-pencil, or else $|S| \leq q^2 - q + q^{\frac{2}{3}} - \frac{2}{3}q^{\frac{1}{3}} + 1$.*
2. *If $q = 4$ then either $|S| = 16 = q^2$ and S is a point-pencil, or else $|S| \leq 13 = q^2 - q + 1$.*
3. *If $q = 3$ then either $|S| = 9 = q^2$ and S is a point-pencil, or else $|S| \leq 8$.*

Corollary

$Aut(\mathcal{U}) \cong Aut(\Gamma_{\mathcal{U}})$.


The End