Group Rings and Geometry: The (FA) Property

Finite Geometry & Friends

Doryan Temmerman

Joint work with A. Bächle, G. Janssens, E. Jespers and A. Kiefer

June 19, 2019

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Geometric Group Theory

EXAMPLE 1

$T =$ 

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$\Downarrow$

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HNN EXTENSION

$$B \leq A \text{ groups}$$

$$f : B \hookrightarrow A$$

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$\Rightarrow \Gamma$ is a HNN extension

AMALGAMATED PRODUCT

$$A, B, C \text{ groups}$$

$$f : C \hookrightarrow A, \qquad g : C \hookrightarrow B$$

$$\Rightarrow A *_C B = \langle A, B \mid \forall c \in C : f(c) = g(c) \rangle$$

Non-trivial if neither f nor g are surjections.

EXAMPLE 2

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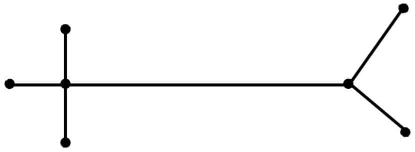

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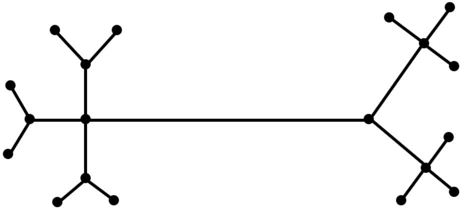

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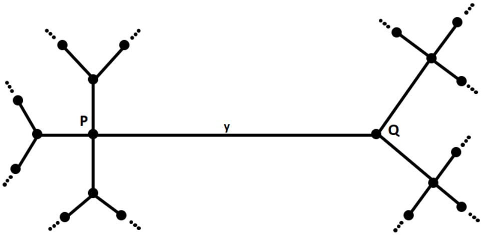

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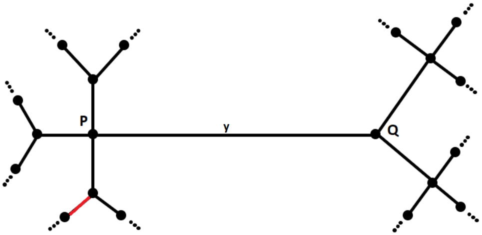

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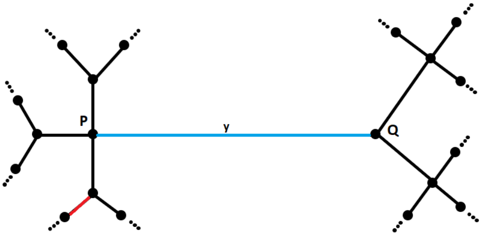

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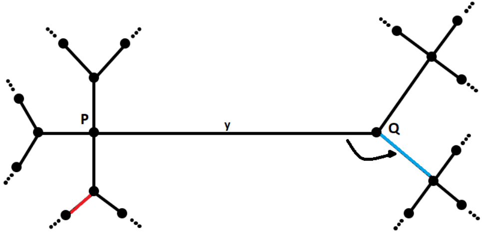

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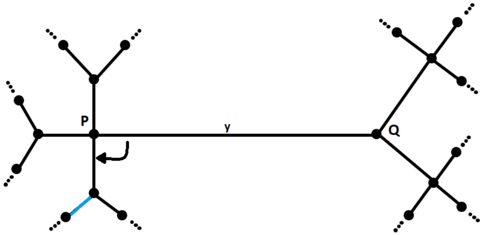

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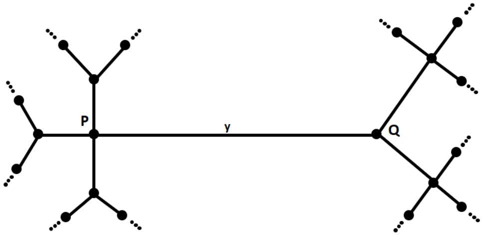

Stabilizer of P is C_4 and the stabilizer of Q is C_3. The stabilizer of y is the trivial group.

AMALGAMATED PRODUCT CONT.

Theorem (Serre ’68)

A group Γ acts on a tree with as fundamental domain 

if and only if there exist groups A, B and C such that $\Gamma \cong A *_C B$. Moreover, in this case, $A \cong \Gamma_P$, $B \cong \Gamma_Q$ and $C \cong \Gamma_y$, the stabilizers in Γ of P, Q and y respectively.

TORSION ELEMENTS AND PROPERTY (FA)

Fact

Torsion elements of $A*_f$ are conjugate to elements of A.
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Definition (Property (FA))

A group Γ is said to have property (FA) if every Γ-action on a tree, without inversion, has a global fix point.

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Lemma (Serre, '68)

For a finitely generated group Γ holds

Γ has property (FA) $\Leftrightarrow$
- *Γ is not a HNN extension*
- *Γ is not an amalgamated product*

Group Rings

WHAT ARE GROUP RINGS?

Definition (Group Ring)

Let $(G, .)$ be a group and $(R, +, .)$ an unital ring. The group ring RG has as additive structure the free R-module on the abstract symbols of G. The multiplication is defined to be the R-linear expansion of the product in the group G.

$$RG = \left\{ \sum_{g \in G} a_g g \mid a_g \in R,\ a_g \neq 0 \text{ for only finitely many } g\text{'s} \right\}$$

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$$\mathbb{Z}G$$

PROJECT

Question

Let G be a finite group. When does $\mathcal{U}(\mathbb{Z}G)$ have (FA)?

THE PROBLEM WITH (FA)...

Fact

Let K be a finite index subgroup of Γ, then

$$K \text{ has (FA)} \Rightarrow \Gamma \text{ has (FA)}$$

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$$\mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right) \quad \text{has (FA)}$$

f.i.

$$\mathrm{SL}_2\left(\mathbb{Z}\left[\sqrt{-3}\right]\right) \quad \text{does not have (FA)}$$

THE SOLUTION

Definition (Property (HFA))

A group Γ is said to have property (HFA) if every finite index subgroup has property (FA).

Fact

Let K be a finite index subgroup of Γ, then

$$K \text{ has (HFA)} \Leftrightarrow \Gamma \text{ has (HFA)}$$

(HFA) INSTEAD OF (FA)

Question

Let G be a finite group. When does $\mathcal{U}(\mathbb{Z}G)$ have (HFA)?

Idea: reduction to special linear groups $\mathrm{SL}_n(\mathcal{O})$ over orders, i.e. a subring of a $\mathbb{Q}$-algebra which is a free $\mathbb{Z}$-module and contains a $\mathbb{Q}$-basis for the algebra.

PROPERTY (HFA) FOR $\mathcal{U}(\mathbb{Z}G)$

Theorem (Bächle-Janssens-Jespers-Kiefer-T.)

$\mathcal{U}(\mathbb{Z}G)$ *has* (HFA) $\Leftrightarrow$ G *is a cut group and does not have an epimorphic image in a specific list of* 10 *groups*

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$\Leftrightarrow$ $\mathcal{U}(\mathbb{Z}G)$ *has Kazhdan's property* (T)

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$\mathcal{U}(\mathbb{Z}G)$ has (HFA) $\Leftrightarrow$ *G is a cut group and does not have an epimorphic image in a specific list of 10 groups*

$\Leftrightarrow$ *$\mathcal{U}(\mathbb{Z}G)$ has Kazhdan's property (T)*

$\Leftrightarrow$ *All finite index subgroups of $\mathcal{U}(\mathbb{Z}G)$ have finite abelianization*