More quasi-symmetric 2-$(56, 16, 6)$ designs *

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Definition.

A t-(v, k, λ) **design** is a set of v points and a collection of k-subsets called blocks, with the property that any t-subset of points is contained in exactly λ blocks.

For a t-(v, k, λ) design we denote by b the **total number of blocks**, and by r the **number of blocks through any point**:

$$b = \lambda \cdot \frac{\binom{v}{t}}{\binom{k}{t}} \qquad r = \lambda \cdot \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$$

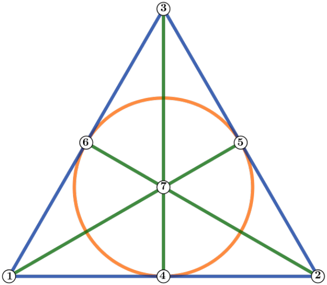
The numbers t, v, k, λ, b and r are **parameters** of the design.

Example: 2-$(7,3,1)$ Fano plane

$\mathcal{V} = \{1,2,3,4,5,6,7\}$
$\mathcal{B} = \{\{1,2,4\},\{2,3,5\},\{1,3,7\},\{1,5,6\},\{3,4,6\},\{2,6,7\},$
$\{4,5,7\}\}$

The total number of blocks: $b = 7$.

The number of blocks through any point: $r = 3$.



Definition.

A t-(v, k, λ) design is **quasi-symmetric** if any two blocks intersect either in x or in y points, for non-negative integers $x < y$.

The numbers x and y are called **intersection numbers**.

- Any symmetric 2-design $(v = b)$ is quasi-symmetric with $x = \lambda$ and y is arbitrary.
- Any Steiner 2-design $(\lambda = 1)$ is quasi-symmetric with $x = 0$ and $y = 1$.

GOAL: construct new quasi-symmetric 2-designs with exceptional parameters

M.S. Shrikhande, *Quasi-symmetric designs*, in: *The Handbook of Combinatorial Designs*, Second Edition (editors: C.J. Colbourn i J.H. Dinitz), CRC Press, 2007., pp. 578–582.

No.	v	k	λ	r	b	x	y	Existence	Ref.
...									
47	56	16	18	66	231	4	8	?	
48	56	15	42	165	616	3	6	?	
49	56	12	9	45	210	0	3	?	
50	56	21	24	66	176	6	9	?	
51	56	20	19	55	154	5	8	?	
52	56	16	6	22	77	4	6	Yes(≥ 2)	[2045, 1659]
...									

No.	v	k	λ	r	b	x	y	Existence	Ref.
...									
47	56	16	18	66	231	4	8	≥ 4	
48	56	15	42	165	616	3	6	0	
49	56	12	9	45	210	0	3	?	
50	56	21	24	66	176	6	9	0	
51	56	20	19	55	154	5	8	?	
52	56	16	6	22	77	4	6	Yes(≥ 2)	[2045, 1659]
...									

V. D. Tonchev, Embedding of the Witt-Mathieu system $S(3, 6, 22)$ in a symmetric 2-$(78, 22, 6)$ design, Geom. Dedicata, 22 (1987), 49–75.

A. Munemasa and V. D. Tonchev, A new quasi-symmetric 2-$(56, 16, 6)$ design obtained from codes, Discrete Math., 284 (2004), 231–234.

We use **computational methods** for the construction of quasi-symmetric designs with prescribed automorphism groups.

METHOD 1: a method based on clique search

METHOD 2: a method based on tactical decompositions

METHOD 3: a method based on binary codes

GAP – Groups, Algorithms, and Programming, Version 4.8.10, 2018.
https://www.gap-system.org

W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system I. The user language*, J. Symbolic Comput., 24(3-4):235–265, 1997.

METHOD 1: a method based on clique search

1: select a **group** G

Let G be a permutation group on a v-element set.

2: compute **good orbits** under G

Let $\mathcal{K}_1, ..., \mathcal{K}_n$ be the **good orbits** of k-element subsets of the v-element set induced by G.

An orbit $\mathcal{K}$ is **good** if

$$|K_1 \cap K_2| = x \text{ or } y,$$

for any two elements $K_1, K_2 \in \mathcal{K}$.

METHOD 1: a method based on clique search

We use our own **C program** to compute good orbits. It is based on an orderly generation algorithm of Read-Faradžev type.

I.A. Faradžev, *Constructive enumeration of combinatorial objects*, Problèmes combinatoires et théorie des graphes, Colloq. Internat. CNRS 260, Paris, 1978, pp. 131-135.

R.C. Read, *Every one a winner or how to avoid isomorphism search when cataloguing combinatorial configurations*, Annals of Discrete Mathematics 2 (1978), 107-120.

METHOD 1: a method based on clique search

3: use cliques algorithm

A **clique** is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent.

Let Γ be the **graph** with following properties:

- vertices are the **good orbits** $\mathcal{K}_1, \ldots, \mathcal{K}_n$,
- two vertices are joined if the corresponding orbits are **compatible**,
- the weight of a vertex is the size of the orbit.

The graph Γ is called the **compatibility graph** of the orbits.

PROBLEM: find all cliques of weight b in the graph Γ

METHOD 1: a method based on clique search

S. Niskanen, P.R.J. Östergård, *Cliquer User's Guide, Version 1.0*, Communications Laboratory, Helsinki University of Technology, Espoo, Finland, Tech. Rep. T48, 2003.

Searching all cliques of a given size in the graph is a NP complete problem.

... it is easier if **the density of the graph** Γ is small.

$$D = \frac{2|E|}{|V|(|V|-1)},$$

where E is the set of edges and V is the set of vertices in the graph Γ.

RESULT: collections of b compatible blocks (not necessary designs)

METHOD 1: a method based on clique search

4: **test** for designs

We need to check the property that any 2-subset of v points is contained in exactly λ blocks.

$$\Downarrow$$

876 new quasi-symmetric 2-$(56, 16, 6)$ designs

METHOD 1: a method based on clique search

Let G_{48} be a certain permutation group on 56 points isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$.

The number of orbits of 16-element subsets: $867\,693\,085\,859$

The number of **good orbits** of 16-element subsets: $5\,352$

The **compatibility graph**: $5\,352$ vertices and $379\,369$ edges (density 0.02649)

The number of **cliques** of weight $b = 77$: $224\,256$

The number of **designs**: $224\,256$

Theorem.

There are 876 quasi-symmetric designs 2-$(56, 16, 6)$, $x = 4$, $y = 6$ with G_{48} as automorphism group.

METHOD 2: a method based on tactical decompositions

1: select a **group** G

2: generate **good orbit matrices**

Let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ and $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be the **point-** and **block-orbits** of a 2-(v, k, λ) design with respect to a group of automorphisms G.

Let $\nu_i = |\mathcal{V}_i|$ and $\beta_i = |\mathcal{B}_i|$: $\sum_{i=1}^{m} \nu_i = v$ and $\sum_{j=1}^{n} \beta_j = b$.

$$\nu = (\nu_1, \ldots, \nu_m) \qquad \beta = (\beta_1, \ldots, \beta_n)$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, let

$$b_{ij} = |\{p \in \mathcal{V}_i \mid p \in B\}|, \qquad \text{for } B \in \mathcal{B}_j.$$

METHOD 2: a method based on tactical decompositions

The number b_{ij} is independent of the choice of $B \in \mathcal{B}_j$.

The matrix $B = [b_{ij}]$ has following properties:

1. $\sum_{i=1}^{m} b_{ij} = k, \qquad \text{for } 1 \leq j \leq n$
2. $\sum_{j=1}^{n} \frac{\beta_j}{\nu_i} b_{ij} = r, \qquad \text{for } 1 \leq i \leq m$
3. $\sum_{j=1}^{n} \frac{\beta_j}{\nu'_i} b_{ij} b_{i'j} = \begin{cases} \lambda\nu_i, & \text{for } i \neq i' \\ \lambda(\nu_i - 1) + r, & \text{for } i = i' \end{cases}, \quad \text{for } 1 \leq i, i' \leq m$

Any matrix with these properties is called an **orbit matrix**.

METHOD 2: a method based on tactical decompositions

For quasi-symmetric designs with intersection numbers x and y the matrix $B = [b_{ij}]$ has further properties (COLUMN TEST):

$$\sum_{i=1}^{m} \frac{\beta_j}{\nu_i} b_{ij} b_{ij'} = \begin{cases} \alpha x + (\beta_j - \alpha) y, & \text{for } j \neq j' \\ \alpha x + (\beta_j - \alpha - 1) y + k, & \text{for } j = j' \end{cases},$$

for $1 \leq j, j' \leq n$, and

$$\alpha = |\{B \in \mathcal{B}_j \mid |\mathcal{B} \cap \mathcal{B}'| = x,\, B' \in \mathcal{B}_{j'}\}|.$$

We shall call any such orbit matrix **good**.

METHOD 2: a method based on tactical decompositions

3: **compute orbits** under G using good orbit matrices

We generate **block orbits compatible with the columns of an good orbit matrix**, i.e. having the prescribed intersection pattern with the point orbits $\mathcal{V}_1, \ldots, \mathcal{V}_m$.

4: use **backtracking solver**

We get fewer block orbits and information on how to choose them to get the desins (one orbit for every column).

We use our **own backtracking program** instead of cliquer to make use of this information.

$\Downarrow$

304 new quasi-symmetric 2-$(56, 16, 6)$ designs

METHOD 2: a method based on tactical decompositions

We consider all possible actions of the group $G_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ on 56 points.

The group $\mathbb{Z}_7.\mathbb{Z}_3$ can act as a permutation group on orbits of size 7 and 21.

Lema.

An automorphism of order 7 of a quasi-symmetric design 2-$(56, 16, 6)$, $x = 4$, $y = 6$ acts without any fixed points and blocks.

ν	β	#OM	#GOM	$\#\mathcal{D}$
$(7,7,7,7,7,7,7,7)$	$(7,7,7,7,7,7,7,7,21)$	26	26	0
$(7,7,7,7,7,21)$	$(7,7,7,7,7,7,7,7,21)$	501	8	0
$(7,7,7,7,7,21)$	$(7,7,7,7,7,21,21)$	8	8	0
$(7,7,21,21)$	$(7,7,7,7,7,21,21)$	16	4	2
$(7,7,21,21)$	$(7,7,21,21,21)$	1	1	0

METHOD 2: a method based on tactical decompositions

$$A_1 = \begin{bmatrix} 4 & 4 & 3 & 1 & 1 & 2 & 1 \\ 3 & 0 & 4 & 3 & 3 & 1 & 2 \\ 6 & 6 & 3 & 9 & 3 & 6 & 7 \\ 3 & 6 & 6 & 3 & 9 & 7 & 6 \end{bmatrix} \quad A_2 = \begin{bmatrix} 4 & 4 & 3 & 1 & 1 & 2 & 1 \\ 3 & 0 & 1 & 3 & 0 & 3 & 2 \\ 6 & 6 & 3 & 3 & 9 & 6 & 7 \\ 3 & 6 & 9 & 9 & 6 & 5 & 6 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 3 & 3 & 3 & 0 & 2 & 1 \\ 0 & 4 & 1 & 1 & 1 & 3 & 2 \\ 6 & 6 & 9 & 3 & 9 & 5 & 6 \\ 6 & 3 & 3 & 9 & 6 & 6 & 7 \end{bmatrix} \quad A_4 = \begin{bmatrix} 4 & 1 & 1 & 1 & 0 & 3 & 2 \\ 0 & 3 & 3 & 0 & 1 & 2 & 3 \\ 6 & 9 & 3 & 9 & 6 & 5 & 6 \\ 6 & 3 & 9 & 6 & 9 & 6 & 5 \end{bmatrix}$$

METHOD 2: a method based on tactical decompositions

$$A_3^{\tau} = \begin{bmatrix} 4 & 0 & 6 & 6 \\ 3 & 4 & 6 & 3 \\ 3 & 1 & 9 & 3 \\ 3 & 1 & 3 & 9 \\ 0 & 1 & 9 & 6 \\ 2 & 3 & 5 & 6 \\ 1 & 2 & 6 & 7 \end{bmatrix} \begin{matrix} \rightarrow & \textbf{882} \text{ good orbits} \\ \rightarrow & \textbf{588} \text{ good orbits} \\ \rightarrow & \textbf{490} \text{ good orbits} \\ \rightarrow & \textbf{490} \text{ good orbits} \\ \rightarrow & \textbf{735} \text{ good orbits} \\ \rightarrow & \textbf{3674412} \text{ good orbits} \\ \rightarrow & \textbf{3628548} \text{ good orbits} \end{matrix}$$

Theorem.

There are two quasi-symmetric designs 2-$(56, 16, 6)$, $x = 4$, $y = 6$ with G_{21} as automorphism group.

METHOD 2: a method based on tactical decompositions

We consider some possible actions of the group $G_{12} \cong A_4$ on 56 points.

The group A_4 can act as a permutation group on orbits of size 3, 4, 6 and 12.

(1) $\nu = (4,4,6,6,6,6,12,12)$
$\beta = (1,1,1,3,3,4,4,6,6,12,12,12,12)$
$\Rightarrow$ 253 **quasi-symmetric designs** (67 **new**)

(2) $\nu = (1,3,4,6,6,12,12,12)$
$\beta = (1,1,3,4,4,4,6,6,6,6,12,12,12)$
$\Rightarrow$ 500 **quasi-symmetric designs** (236 **new**)

Theorem.

There are at least 753 quasi-symmetric designs 2-$(56,16,6)$, $x = 4$, $y = 6$ with G_{12} as automorphism group.

METHOD 3: a method based on binary codes

1: generate a **binary code**

Let C be the binary code spanned by block incidence vectors of quasi-symmetric 2-(v, k, λ) design with intersection numbers x and y.

2: identify **codewords of weight k into orbits** (optional)

We can identify codewords of code C of weight k into orbits under various automorphism groups G.

METHOD 3: a method based on binary codes

3: use **cliques algorithm**

Let Γ be the **graph** with following properties:

- vertices are the (orbits of) **codewords of weight** k,
- two vertices are joined if the corresponding (orbits of) codewords are **compatible**,
- weight of a vertex is equal to 1 (or size of the orbit).

The graph Γ is called the **compatibility graph** of the orbits.

PROBLEM: find all cliques of size (weight) b in the graph Γ

4: **test** for designs

$\Downarrow$

228 new quasi-symmetric 2-$(56, 16, 6)$ designs

METHOD 3: a method based on binary codes

The 1182 known 2-$(56, 16, 6)$ designs span 39 **inequivalent codes** $C_1, \ldots, C_{39}$.

	dim	a_0	a_8	a_{12}	a_{16}	a_{20}	a_{24}	a_{28}
C_1	26	1	91	2016	152425	2939776	16194619	28531008
C_2	26	1	7	2016	155365	2926336	16224019	28493376
C_3	24	1	75	0	40089	730368	4055835	7124480
$C_{4\text{-}6,9,10}$	22	1	15	0	9933	183168	1012515	1783040
$C_{7,11\text{-}13}$	25	1	75	672	77721	1465984	8103963	14257600
C_8	25	1	75	960	75417	1474048	8087835	14277760
C_{14}	22	1	15	0	10701	178560	1024035	1767680
C_{15}	23	1	15	288	19917	361216	2040867	3544000
C_{16}	23	1	15	96	19917	365056	2028579	3561280
C_{17}	24	1	75	160	39833	728704	4062235	7115200
C_{18}	22	1	15	64	9677	183424	1012771	1782400
$C_{19,21,24}$	22	1	15	16	10061	182080	1015459	1779040
$C_{20,22}$	22	1	15	64	10445	178816	1024291	1767040
C_{23}	25	1	75	1280	74905	1470720	8100635	14259200
C_{25}	25	1	75	992	77209	1462656	8116763	14239040
C_{26}	27	1	139	4992	307161	5848832	32477083	56941312
C_{27}	27	1	99	4304	305873	5872320	32406731	57039072
$C_{28,29}$	27	1	99	4112	307409	5866944	32417483	57025632
C_{30}	26	1	147	1008	158529	2920512	16231467	28485536
$C_{31,32,34,35}$	27	1	147	3696	309057	5862976	32423979	57018016
$C_{33,39}$	27	1	147	4976	307009	5849664	32475179	56943776
C_{36}	26	1	75	2240	153241	2931200	16218395	28498560
$C_{37,38}$	27	1	75	4416	305817	5871616	32408859	57036160

METHOD 3: a method based on binary codes

Let G_{16} be a certain permutation group on 56 points isomorphic to $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 \rtimes \mathbb{Z}_4 \rtimes \mathbb{Z}_4$.

We identify **codewords of weight** 16 of codes $C_3, \ldots, C_{25}$ **into orbits** under the group G_{16}, and use **clique algorithm** to find all cliques of weight $b = 77$ in the compatibility graphs.

Theorem.

There are at least 228 quasi-symmetric designs 2-$(56, 16, 6)$, $x = 4$, $y = 6$ with G_{16} as automorphism group.

Theorem.

There are at least $1\,410$ quasi-symmetric designs 2-$(56,16,6)$, $x=4$, $y=6$.

$\lvert\mathrm{Aut}\rvert$	$\#(56,16,6)$
168	1
48	876
24	1
21	1
16	228
12	303

Thank you for your attention!