Directed regular graphs from groups

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Content

Definition

A **simple graph** $\mathcal{G}$ *consists of a non-empty finite set* $\mathcal{V}(\mathcal{G})$, *whose elements are called vertices and a finite set* $\mathcal{E}(\mathcal{G})$ *of different 2-subsets of set* $\mathcal{V}(\mathcal{G})$ *whose elements are called edges.*

Definition

A **directed graph** *or a* **digraph** $\mathcal{G}$ *consists of a non-empty finite set* $\mathcal{V}(\mathcal{G})$, *whose elements are called vertices, and of a finite family* $\mathcal{E}(\mathcal{G})$ *of ordered pairs of elements of set* $\mathcal{V}(\mathcal{G})$ *whose elements are called arcs.*

Definition

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a graph with n vertices. Graph $\mathcal{G}$ is **strongly regular graph** *or* **SRG** *with parameters (n, k, λ, μ), SRG(n, k, λ, μ), if*

1. *$\mathcal{G}$ is simple k-regular graph,*
2. *any two adjacent vertices have λ common neighbours,*
3. *any two non-adjacent vertices have μ common neighbours.*

Definition

A **quasi-strongly regular graph** *$(QSRG)$ with parameters $(n, k, a; c_1, c_2, \ldots, c_p)$ is a k-regular graph on n vertices such that any two adjacent vertices have a common neighbours and any two non-adjacent vertices have c_i common neighbours for some $1 \leq i \leq p$.*

Art M. Duval, *A directed graph version of strongly regular graphs*, Journal of Combinatorial Theory, 1988.

Definition

A **directed strongly regular graph** *with parameters* (n, k, λ, μ, t) *is a directed graph* Γ *on* n *vertices without loops such that*

(i) *every vertex has in-degree and out-degree* k,

(ii) *every vertex* x *has* t *out-neighbours that are also in-neighbours of* x, *and*

(iii) *the number of directed paths of length 2 from a vertex* x *to another vertex* y *is* λ *if there is an edge from* x *to* y, *and is* μ *if there is no edge from* x *to* y.

Such a graph Γ *is called a DSRG*(n, k, λ, μ, t).

The adjacency matrix $A = A(\Gamma)$ of directed strongly regular graph satisfies

$$A^2 = tI + \lambda A + \mu(J - I - A),$$
$$AJ = JA = kJ.$$

DSRG with $t = k$ is SRG.

Guo, Z., Jia, D. and Zhang, G. *Some Constructions of Quasi-strongly Regular Digraphs*. Graphs and Combinatorics, 2022.

Definition

A **quasi-strongly regular digraph** *with parameters* $(n, k, t, a; c_1, c_2, \ldots, c_p)$*, also denoted by* $QSRD(n, k, t, a; c_1, c_2, \ldots, c_p)$*, is a* k*-regular digraph on* n *vertices such that*

(i) *each vertex is incident to* t *undirected edges;*

(ii) *for any two vertices* $x \to y$ *the number of paths of length 2 from* x *to* y *is* a*;*

(iii) *for any distinct vertices* $x \nrightarrow y$ *the number of paths of length 2 from* x *to* y *is* c_i*, where* $1 \leq i \leq p$*,*

(iv) *for any* $1 \leq i \leq p$ *there exist distinct vertices* $x \nrightarrow y$ *such that the number of paths of length 2 from* x *to* y *is* c_i*.*

Proposition

Let Γ *be a digraph with* n *vertices and let* A *be the adjacency matrix of* Γ*. Then* Γ *is a* $QSRD(n, k, t, a; c_1, c_2, \ldots, c_p)$ *if and only if*

$$AJ = JA = kJ,$$
$$A^2 = tI + aA + c_1 C_1 + c_2 C_2 + \cdots + c_p C_p$$

for some non-zero $(0,1)-$*matrices* $C_1, C_2, \ldots, C_p$ *such that* $C_1 + C_2 + \cdots + C_p = J - I - A$*.*

p is the grade of QSRD.

Definition

*Let Ω be a finite set and let $R_0, R_1, \ldots, R_d$ be a partition of $\Omega \times \Omega$. Then $(\Omega, \{R_0, R_1, \ldots, R_d\})$ is called a d-***class association scheme** *if the following conditions hold.*

(i) $R_0 = \{(x,x) | x \in \Omega\}$;

(ii) *for any* $i \in \{0, 1 \ldots, d\}$, *there exists* $i' \in \{0, 1, \ldots, d\}$, *such that*

$$R_{i'} = \{(x,y) | (y,x) \in R_i\};$$

(iii) *for any* $i, j, k \in \{0, 1, \ldots, d\}$ *and any pair* $(x,y) \in R_k$, *the number*

$$p_{ij}{}^k = |\{z \in \Omega | (x,z) \in R_i \text{ and } (z,y) \in R_j\}|$$

depends only on i, j, k.

Theorem

Let $(\Omega, \{R_0, R_1, R_2, \ldots, R_d\})$ be a d-class association scheme, and let Γ_i be a digraph with vertex set Ω and arc set R_i, where $i \in \{1, 2, \ldots, d\}$. Then each Γ_i is a QSRD. Moreover, the grade of Γ_i is p if and only if $p_{ii}{}^j$ takes on p distinct values as j ranges over $\{1, 2, \ldots, d\} \backslash \{i\}$.

Definition

*A group G **acts** on a set Ω if there exists a function $f : G \times \Omega \to \Omega$ such that*

1. $f(e, x) = x$, $\forall x \in \Omega$,
2. $f(g_1, f(g_2, x)) = f(g_1 g_2, x)$, $\forall x \in \Omega$, $\forall g_1, g_2 \in G$.

Denote the described action by $g.x$, $g \in G$.
The set $G_x = \{g \in G | g.x = x\}$ is a group called **stabilizer** of the element $x \in \Omega$.
The action of the group G on set Ω induces the equivalence relation on set Ω: $x \sim y \Leftrightarrow (\exists g \in G)\, g.x = y$. The equivalence classes are **orbits** of the action.

Definition

Group G acts **transitively** *on set Ω if*

$$(\forall x, y \in \Omega)(\exists g \in G) \text{ such that } g.x = y,$$

that is, if there exists an element $x \in \Omega$ such that $G.x = \Omega$.

Let group G act transitively on set Ω. Then group G acts on set $\Omega \times \Omega$ like this: $g.(x_1, x_2) = (g.x_1, g.x_2)$. Orbits for that action are called **orbitals** of group G on set Ω.

G transitive permutation group on set Ω, $H \leq G$.

- For each orbital Δ there is an orbital Δ^*, where $(\alpha, \beta) \in \Delta^*$ if and only if $(\beta, \alpha) \in \Delta$. An orbital is *self-paired* if $\Delta^* = \Delta$.
- $T \subseteq G$ is a **left (right) transversal** or a set of representatives of all left (right) cosets of H in G if T contains exactly one element of each left (right) coset aH (Ha), $a \in G$.
- There exists a bijection from the set of orbitals to set of G_α-orbits. G_α-orbits are called **suborbits**, and their sizes are **subdegrees** of permutation group G.

Construction of transitive 1-designs from finite group:
D. Crnković, V. Mikulić Crnković and A. Švob: *On some transitive combinatorial structures constructed from the unitary group $U(3,3)$.* Journal of Statistical Planning and Inference, 2014.

Theorem

Let G be a finite permutation group acting transitively on sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \cup_{i=1}^{s} G_\alpha.\delta_i$, where $\delta_1, \ldots, \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{g.\Delta_2 : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s) = (\Omega_2, \mathcal{B})$ is a 1-$(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |G_{\delta_i}.\alpha|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \cap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design.

Construction of directed regular graphs:

Theorem (MZ, Mikulić Crnković, Vedrana)

Let G be a finite permutation group acting transitively on the set Ω. Let $\alpha \in \Omega$ and let $\Delta = \cup_{i=1}^{s} \delta_i G_\alpha$ be a union of orbits of the stabilizer G_α of α, where $\delta_1, \ldots, \delta_s$ are representatives of different G_α- orbits. Let $T = \{g_1, \ldots, g_t\}$ be a set of representatives of left cosets in $G/G_\alpha = \{g_1 G_\alpha, \ldots, G_t G_\alpha\}$. Let $\mathcal{V} = \{g_i.\alpha | i = 1, \ldots, t\}$ and let $\mathcal{E} = \{(g_i.\alpha, g_i.\beta) | i = 1, \ldots, t, \beta \in \Delta\}$. Then $\Gamma = (\mathcal{V}, \mathcal{E})$ is a directed graph with $|\Omega|$ vertices that is $|\Delta|$-regular and such that $g_i.\Delta$ is a set of out-neighbours of the vertex $g_i.\alpha$, $i = 1, \ldots, t$.

Theorem (MZ, Mikulić Crnković, Vedrana)

If a group G acts transitively on a set of vertices of a directed regular graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then there exists a set Ω such that vertices and arcs of a digraph $\mathcal{G}$ are defined in the way described in the previous theorem.

Example

D_3 acts transitively on $\Omega = \{1,2,3,4,5,6\}$ in six suborbits of length 1.
Take $\Delta_1 \cup \Delta_2 = \{2,3\}$ as a set of out-neighbours of vertex $\alpha = 1 \in \Omega$:

$$g_1.\{2,3\} = \{2,3\} \text{ is a set of out-neighbours of a vertex } g_1.\alpha = 1,$$
$$g_2.\{2,3\} = \{1,6\} \text{ is a set of out-neighbours of a vertex } g_1.\alpha = 2,$$
$$g_3.\{2,3\} = \{4,5\} \text{ is a set of out-neighbours of a vertex } g_1.\alpha = 3,$$
$$g_4.\{2,3\} = \{3,2\} \text{ is a set of out-neighbours of a vertex } g_1.\alpha = 4,$$
$$g_5.\{2,3\} = \{6,1\} \text{ is a set of out-neighbours of a vertex } g_1.\alpha = 5,$$
$$g_6.\{2,3\} = \{5,4\} \text{ is a set of out-neighbours of a vertex } g_1.\alpha = 6.$$

Adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

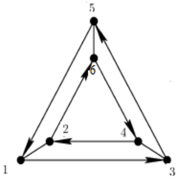

Figure: *DSRG(6,2,0,1,1)*

GAP - Transitive Groups Library:

Theorem

1. *There are, up to isomorphism, 478 quasi-strongly regular graphs on which a transitive automorphism group of degree n,* $n \in \{4, \ldots, 30\} \backslash \{22, 24, 28, 30\}$, *acts. 19 of them are strongly regular graphs.*
2. *There are, up to isomorphism, 2920 directed quasi-strongly regular graphs on which a transitive automorphism group of degree n,* $n \in \{4, \ldots, 30\} \backslash \{22, 24, 28, 30\}$, *acts. 478 of them are directed strongly regular graphs.*

Degree	# **QSRG**	# **SRG**	# **QSRD**	# **DSRG**
22	39	2	18	
24	7853		68171	64
28	213	2	447	22
30	110	40	642	

Table: Number of graphs obtained from transitive non-regular permutation groups of degree n, $n \in \{22, 24, 28, 30\}$

https://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html

v	k	t	λ	μ	r^f	s^g	comments		
21	6	2	1	2	0^{14}	-1^{6}	T8(i) for 2-(7,3,1)	T12	T18 for (d,l,s)=(1,2,3)
	14	10	9	10	0^{6}	-1^{14}	T18 for (d,l,s)=(2,4,3)		
21	8	4	3	3	1^{6}	-1^{14}	T8(ii) for 2-(7,3,1)	T9 for pg(3,3,3)	T18 for (d,l,s)=(1,2,3)
	12	8	6	8	0^{14}	-2^{6}	T12	T18 for (d,l,s)=(2,4,3)	
22	9	6	3	4	1^{11}	-2^{10}	?		
	12	9	6	7	1^{10}	-2^{11}	?		

Figure: (Non)existence of DSRGs with parameters $(22, 9, 3, 4, 6)$ and $(22, 12, 6, 8, 8)$

Graphs from unions of length $k = 9$ and $k = 12$ from regular permutation representations of $\mathbb{Z}_{22}$ and D_{11} of degree 22:

Degree	Parameters	# non-isom.	Aut($\mathcal{G}$)
22	QSRG(22,9,0;8,7,0)	1	D_{22}
	QSRD(22,9,5,4;4,3)	1	D_{22}
	QSRD(22,9,7,0;9,8,7,0)	4	$\mathbb{Z}_{22}$
	QSRD(22,9,8,0;9,8,7,0)	1	$\mathbb{Z}_{22}$

Table: Graphs obtained from regular permutation groups $\mathbb{Z}_{22}$ and D_{11} of degree 22

Future work...

- Construction of self-orthogonal codes from adjacency matrix A of $\mathrm{DSRG}(n, k, \lambda, \mu, t)$ with $t = \mu$.
- Construction of LCD codes from matrices $[A|I_n]$ and $[A, I_n, \mathbb{1}]$, where A is the adjacency matrix of $\mathrm{DSRG}(n, k, \lambda, \mu, t)$ with $t = \mu$.

Thank you for your attention!