ABOUT NUMBER THEORY AND AKSHAY VENKATESH

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The work of Akshay Venkatesh is characterised by the use of diverse and often unusual or unexpected tools from dynamics, topology and representation theory in number theory. In this talk we will present some examples of his innovations in the advancement of classical and newer arithmetical settings.

In particular we will encounter the Riemann zeta function, its generalisation to automorphic L-functions and the long-standing conjectures on their zeros which when proved will give information on the distribution of prime numbers. An equivalent formulation is the study of the bounds of these functions on the critical line where the zeros are expected to lie. The subconvexity problem, begun by Weyl using exponential sums, is to beat convexity bounds available from the functional equation.

Venkatesh used ergodic theoretic techniques for equidistribution of orbits in certain homogeneous spaces to obtain subconvexity for L-functions in great generality and resolved the problem completely with Michel in 2000 for GL_2. Suitable subconvexity bounds lead to insights on many other topics like the representation of integers by integral ternary quadratic forms, answering questions of the type 'how can a large integer be written as the sum of three squares?'.

In fact the question of when one integral quadratic form in m variables represents another in $n \leq m$ variables is a variant of Hilbert's 11th problem and has a long and illustrious history beginning with Siegel. Using the dynamics of lattices, Venkatesh with Ellenberg made an astonishing contribution by proving a local-global principle for $n \geq m + 5$ which brings m and n much closer than had been expected. This work is a follow-up of Linnik's on $m = 1$ but the group associated to the quadratic form is no longer a torus and they now need powerful results on the flow of lattices.

Another contribution came from the use of algebraic topology and algebraic geometry to study the class number of a function field. The original Cohen-Lenstra heuristics conjectured that the distribution of class groups of imaginary quadratic extensions of the rational numbers were governed by a probabilistic law on finite abelian groups. Though the computational evidence supporting this is large, rigorous proofs are missing. Venkatesh and his co-authors, Ellenberg and Westerland, very recently proved that these heuristics were exactly as predicted for quadratic extensions of rational functions over a finite field. To arrive at this they used the notion of homological stability for a class of topological objects called Hurwitz spaces which are not even connected.

If time permits we will mention other contributions of Venkatesh, in particular, his on-going work on the Langlands programme .